

# A priori Estimates for the Compressible Euler Equations for a Liquid with Free Surface Boundary and the Incompressible Limit

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## Abstract

In this paper, we prove a new type of energy estimates for the compressible Euler's equation with free boundary, with a boundary part and an interior part. These can be thought of as a generalization of the energies in Christodoulou and Lindblad [1] to the compressible case and do not require the fluid to be irrotational. In addition, we show that our estimates are in fact uniform in the sound speed  $\kappa$ . As a consequence, we obtain convergence of solutions of compressible Euler equations with a free boundary to solutions of the incompressible equations, generalizing the result of Ebin [6] to when you have a free boundary. In the incompressible case our energies reduces to those in [1] and our proof in particular gives a simplified proof of the estimates in [1] with improved error estimates. Since for an incompressible irrotational liquid with free surface there are small data global existence results our result leaves open the possibility of long time existence also for slightly compressible liquids with a free surface.

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## 1 Introduction

We consider Euler equations

$$\begin{cases} (\partial_t + v^k \partial_k) v = -\rho^{-1} \partial p & \text{in } \mathcal{D}, \\ (\partial_t + v^k \partial_k) \rho + \rho \operatorname{div} v = 0 & \text{in } \mathcal{D}. \end{cases} \quad (1.1)$$

describing the motion of a perfect compressible fluid in vacuum, where  $\partial_i = \frac{\partial}{\partial x_i}$ ,  $\operatorname{div} v = \partial_k v^k$ , and  $v = (v_1, \dots, v_n)$  and  $\mathcal{D} = [0, T] \times \mathcal{D}_t \subset [0, T] \times \mathbb{R}^n$ , and the density  $\rho$  are to be determined. Here,  $v^k = \delta^{ij} v_j = v_k$ , and we have used the summation convention on repeated upper and lower indices. The pressure  $p = p(\rho)$  is assumed to be a given strictly increasing smooth function of the density. The boundary  $\partial \mathcal{D}_t$  moves with the velocity of the fluid particles at the boundary. The fluid moves in the vacuum so the pressure  $p$  vanishes in the exterior and hence on the boundary. We therefore require the boundary condition on  $\partial \mathcal{D} = [0, T] \times \partial \mathcal{D}_t$  to be

$$\begin{cases} (\partial_t + v^k \partial_k)|_{\partial \mathcal{D}} \in T(\partial \mathcal{D}), \\ p = 0 \quad \text{on } \partial \mathcal{D}. \end{cases} \quad (1.2)$$

Since the pressure is assumed to be a strictly increasing function of the density, we can alternatively think the density as a function of the pressure and for physical reasons this function has to be non-negative. Therefore the density has to be a constant  $\bar{\rho}_0 \geq 0$  on the boundary, and we assume in fact  $\bar{\rho}_0 > 0$ , which is in the case of a liquid. Hence,

$$p(\bar{\rho}_0) = 0, \quad p'(\rho) > 0, \quad \text{for } \rho \geq \bar{\rho}_0. \quad (1.3)$$

For the sake of simplicity and in order to make some of our calculations easier, we assume further that

$$\bar{\rho}_0 = 1. \quad (1.4)$$

By thinking of the density as a function of the pressure the incompressible case can be thought as a special case of constant density function.

Given a bounded domain  $\mathcal{D}_0 \subset \mathbb{R}^n$ , that is homeomorphic to the unit ball, and initial data  $v_0$  and  $\rho_0$ , we want to find a set  $\mathcal{D} \in [0, T] \times \mathbb{R}^n$ , a vector field  $v$  and a function  $\rho$ , solving (1.1)-(1.2) and satisfying the initial conditions

$$\begin{cases} \{x : (0, x) \in \mathcal{D}\} = \mathcal{D}_0, \\ v = v_0, \rho = \rho_0 \quad \text{on } \{0\} \times \mathcal{D}_0. \end{cases} \quad (1.5)$$

## 1.1 Enthalpy form

Let  $D_t = \partial_t + v^k \partial_k$  be the material derivative. We introduce the enthalpy  $h$  to be a strictly increasing function of the density, i.e.,  $h(\rho) = \int_1^\rho p'(\lambda) \lambda^{-1} d\lambda$ . Since  $\rho \geq \bar{\rho}_0 = 1$  can then be thought as a function of  $h$ , we define  $e(h) = \log \rho(h)$ . Under these new variables, (1.1)-(1.5) can be re-expressed as

$$\begin{cases} D_t v = -\partial h \quad \text{in } \mathcal{D}, \\ \operatorname{div} v = -D_t e(h) = -e'(h) D_t h \quad \text{in } \mathcal{D}. \end{cases} \quad (1.6)$$

Together with initial and boundary conditions

$$\begin{cases} \{x : (0, x) \in \mathcal{D}\} = \mathcal{D}_0, & \begin{cases} D_t|_{\partial \mathcal{D}} \in T(\partial \mathcal{D}), \\ h = 0 \quad \text{on } \partial \mathcal{D}, \end{cases} \\ v = v_0, h = h_0 \quad \text{on } \{0\} \times \mathcal{D}_0, \end{cases} \quad (1.7)$$

(1.6) looks exactly like the incompressible Euler's equations, where  $h$  takes the position of  $p$  and  $\operatorname{div} v$  is no longer 0 but determined by  $h$ . On the other hand, we would like to impose the following natural conditions on  $e(h)$

1. We assume  $|e^{(k)}(h)| \leq c_0$  for  $k \leq 6$ , for some fixed constant  $c_0$ .
2.  $|e^{(k)}(h)| \leq c_0 \sqrt{e'(h)}$ , for  $k \leq 6$ .

In order for the initial boundary problem (1.6)-(1.7) to be solvable the initial data has to satisfy certain compatibility conditions at the boundary. Bt the second equation in (1.1),(1.3) implies that  $\operatorname{div} v|_{\partial \mathcal{D}} = 0$ . We must therefore have  $h_0|_{\partial \mathcal{D}_0} = 0$  and  $\operatorname{div} v_0|_{\partial \mathcal{D}_0} = 0$ , which is the zero-th compatibility condition. Furthermore,  $m$ -th order compatibility condition can be expressed as

$$(\partial_t + v^k \partial_k)^j h|_{\{0\} \times \partial \mathcal{D}_0} = 0, \quad j = 0, \dots, m. \quad (1.8)$$

Let  $N$  be the exterior unit normal to the free surface  $\partial \mathcal{D}_t$ . We will prove a priori bounds for (1.6)-(1.7) in Sobolev spaces under the assumption

$$\nabla_N h \leq -\epsilon < 0, \quad \text{on } \partial \mathcal{D}_t, \quad (1.9)$$

where  $\nabla_N = N^i \partial_i$  and  $\epsilon > 0$  is a constant. (1.9) is a natural physical condition. It says that the pressure and hence the enthalpy is larger in the interior than at the boundary. The system (1.6)-(1.7) is ill-posed in absence of (1.9), an easy counter-example can be found in [1].

## 1.2 History and background

Euler equations involving free-boundary has been studied intensively by many authors. The first break through in solving the well-posedness for the incompressible and irrotational problem for general data came in the work of Wu [15, 16] who solved the problem in both two and three dimensions. For the general incompressible problem with nonvanishing curl Christodoulou and Lindblad [1] were the first to obtain the energy estimates assuming the Taylor sign condition. For the compressible problem, Lindblad [13] later proved local well-posedness for the general problem modelling the motion of a liquid via Nash-Moser iteration. On the other hand, Coutand-Lindblad-Shkoller [4] and Jang-Masmoudi [12] obtained the energy estimates and well-posedness for the general problem modelling the motion of gas. It is worth mentioning that D. Ebin [6], and Ebin-Disconzi [5] proved the solutions of the compressible equations converges to the solutions of the incompressible equation in Sobolev norms as the sound speed goes to infinity, but within a domain with fixed boundary. But no previous incompressible limit result involving free boundary is known. Our result allows us to approximate slightly compressible liquid by the incompressible liquid in both 2D and 3D, for which global (in time) solution is known to exist (e.g. [8, 9, 10, 11, 17, 18]).

In this paper, we generalize the method used by Chistodoulou and Lindblad [1]. In our proof,  $\operatorname{curl} v$  appears to be of lower orders. In addition, our method is regardless of spatial dimensions. The energy constructed in this paper contains interior and boundary parts, where the interior part controls the velocity and the enthalpy in Sobolev norms. The boundary part contains projected spatial derivatives, which controls the second fundamental form of the moving boundary. The use of projected derivatives on the boundary is crucial due to the loss of regularity when estimating on the boundary, i.e., the trace theorem [7], and the use of the tangential part of derivatives on the boundary compensates the loss.

## 1.3 Energy conservation and higher order energies

The boundary condition  $p|_{\partial \Omega} = 0$  leads to that the zero-th order energy is conserved, i.e., let

$$E_0(t) = \frac{1}{2} \int_{\mathcal{D}_t} \rho |v|^2 dx + \int_{\mathcal{D}_t} \rho Q(\rho) dx, \quad (1.10)$$

where  $Q(\rho) = \int_1^\rho p(\lambda) \lambda^{-2} d\lambda$ . A direct computation yields

$$\begin{aligned} \frac{d}{dt} E_0(t) &= \int_{\mathcal{D}_t} \rho D_t v \cdot v \, dx + \int_{\mathcal{D}_t} p(\rho) D_t \rho \rho^{-1} \, dx = - \int_{\mathcal{D}_t} \partial p \cdot v \, dx + \int_{\mathcal{D}_t} p(\rho) D_t \rho \rho^{-1} \, dx \\ &= - \int_{\mathcal{D}_t} p \operatorname{div} v \, dx + \int_{\mathcal{D}_t} p(\rho) D_t \rho \rho^{-1} \, dx = 0. \end{aligned} \quad (1.11)$$

In order to define higher order energies we introduce a positive definite quadratic form  $Q$  on  $(0, r)$  tensors, which, when restricted to the boundary, is the inner product of the tangential components, i.e.,

$$Q(\alpha, \beta) = \langle \Pi \alpha, \Pi \beta \rangle, \quad \text{on } \partial \mathcal{D}_t, \quad (1.12)$$

where the projection of a  $(0, r)$  tensor to the boundary is defined by

$$(\Pi \alpha)_{i_1, \dots, i_r} = \gamma_{i_1}^{j_1} \cdots \gamma_{i_r}^{j_r} \alpha_{j_1, \dots, j_r}, \quad \text{where } \gamma_i^j = \delta_i^j - \mathcal{N}_i \mathcal{N}^j,$$

and  $\mathcal{N}$  is the unit normal to  $\partial \mathcal{D}_t$ . To be more specific, in the interior we define

$$Q(\alpha, \beta) = q^{i_1 j_1} \cdots q^{i_r j_r} \alpha_{i_1 \dots i_r} \beta_{j_1 \dots j_r}, \quad (1.13)$$

where

$$\begin{aligned} q^{ij} &= \delta^{ij} - \eta(d)^2 \mathcal{N}^i \mathcal{N}^j, \\ d(x) &= \operatorname{dist}(x, \partial \mathcal{D}_t), \\ \mathcal{N}^i &= -\delta^{ij} \partial_j d. \end{aligned}$$

Here  $\eta$  is a smooth cut-off function satisfying  $0 \leq \eta(d) \leq 1$ ,  $\eta(d) = 1$  when  $d \leq \frac{d_0}{4}$  and  $\eta(d) = 0$  when  $d > \frac{d_0}{2}$ .  $d_0$  is a fixed number that is smaller than the injective radius of the normal exponential map  $l_0$ , defined to be the largest number  $l_0$  such that the map

$$\partial \mathcal{D}_t \times (-l_0, l_0) \rightarrow \{x : \operatorname{dist}(x, \partial \mathcal{D}_t) < l_0\}, \quad (1.14)$$

given by

$$(\bar{x}, l) \rightarrow x = \bar{x} + l \mathcal{N}(\bar{x}), \quad (1.15)$$

is an injection.

The higher order energies we propose are

$$E_r = \sum_{s+k=r} E_{s,k} + K_r + W_{r+1}^2, \quad E_r^* = \sum_{r' \leq r} E_{r'}, \quad (1.16)$$

where

$$\begin{aligned} E_{s,k}(t) &= \frac{1}{2} \int_{\mathcal{D}_t} \rho \delta^{ij} Q(\partial^s D_t^k v_i, \partial^s D_t^k v_j) \, dx + \frac{1}{2} \int_{\mathcal{D}_t} \rho e'(h) Q(\partial^s D_t^k h, \partial^s D_t^k h) \, dx \\ &\quad + \frac{1}{2} \int_{\partial \mathcal{D}_t} \rho Q(\partial^s D_t^k h, \partial^s D_t^k h) \nu \, dS, \end{aligned} \quad (1.17)$$

where  $\nu = (-\nabla_N h)^{-1}$  and

$$K_r(t) = \int_{\mathcal{D}_t} \rho |\partial^{r-1} \text{curl } v|^2 dx, \quad (1.18)$$

$$W_r(t) = \frac{1}{2} \|\sqrt{e'(h)} D_t^r h\|_{L^2(\mathcal{D}_t)}^2 + \frac{1}{2} \|\nabla D_t^{r-1} h\|_{L^2(\mathcal{D}_t)}^2. \quad (1.19)$$

Here  $W_r$  is the (higher order) energy for the wave equation

$$D_t^2 e(h) - \Delta h = (\partial_i v^j)(\partial_j v^i), \quad (1.20)$$

which is obtained by commuting divergence through the first equation of (1.1) using

$$[D_t, \partial_i] = -(\partial_i v^j) \partial_j. \quad (1.21)$$

Similarly it follows that we have a transport equation for the curl

$$D_t \text{curl}_{ij} v = -(\partial_i v^k)(\text{curl}_{kj} v) + (\partial_j v^k)(\text{curl}_{ki} v). \quad (1.22)$$

Although the energies  $E_r$  only control the tangential components, the fact that we also control the divergence  $W_{r+1}^2$  (through  $\text{div } v = -D_t e(h)$ ) and the curl  $K_r$  allows us to control all components. In fact, by a Hodge type decomposition

$$|\partial v| \lesssim |\bar{\partial} v| + |\text{div } v| + |\text{curl } v|, \quad (1.23)$$

where the tangential derivatives are given by  $\bar{\partial} h = \Pi \partial h$ .

The boundary term in (1.17) and  $\nu$  are constructed to exactly cancel a boundary term coming from integration by parts in the interior as in (1.11), as will be explained in section 1.5. Moreover the projection in the boundary term is needed to make it lower order in space derivatives of  $h$ . In fact, since  $h$  vanishes on the boundary so does the tangential derivatives  $\bar{\partial} h = \Pi \partial h$  and similarly  $\Pi \partial^r h = O(\partial^{r-1} h)$  is lower order.

Moreover if  $|\nabla_N h| \geq \epsilon > 0$  then the boundary term gives an estimate for the regularity of the boundary. In fact, one can show that if  $q$  vanishes on the boundary then

$$\Pi \partial^r q = (\bar{\partial}^{r-2} \theta) \nabla_N q + O(\partial^{r-1} q) + O(\bar{\partial}^{r-3} \theta), \quad (1.24)$$

where  $\theta$  is the second fundamental form of the boundary and  $\bar{\partial}$  stand for tangential derivatives, so

$$\|\bar{\partial}^{r-2} \theta\|_{L^2(\partial \mathcal{D}_t)}^2 \leq \frac{C}{\epsilon} E_r^* + C \sum_{r' \leq r-1} \|\partial^{r'} h\|_{L^2(\partial \mathcal{D}_t)}^2. \quad (1.25)$$

Because of the bound on the second fundamental form energies in fact control all components

$$\begin{aligned} \|v\|_{r,0} &:= \sum_{k+s=r, k < r} \|\partial^s D_t^k v\|_{L^2(\mathcal{D}_t)}, \\ \|h\|_r &:= \sum_{k+s=r, k < r} \|\partial^s D_t^k h\|_{L^2(\mathcal{D}_t)} + \|\sqrt{e'(h)} D_t^r h\|_{L^2(\mathcal{D}_t)}, \\ \langle \langle h \rangle \rangle_r &:= \sum_{k+s=r} \|\partial^s D_t^k h\|_{L^2(\partial \mathcal{D}_t)}, \end{aligned}$$

in the interior and on the boundary. Using elliptic estimates (see section 3) one can show that

$$\|v\|_{r,0}^2 + \|h\|_r^2 \leq C(K, M, c_0, \text{vol } \mathcal{D}_t) E_r^*, \quad (1.26)$$

$$\|D_t h\|_r^2 + \langle \langle h \rangle \rangle_r^2 \leq C(K, M, c_0, \frac{1}{\epsilon}, \text{vol } \mathcal{D}_t, E_{r-1}^*) E_r^*. \quad (1.27)$$

### 1.4 The main results

We prove energy estimates implying that the higher order energies remain bounded as long as certain a priori assumptions are true (Proposition 5.1). More specifically, let  $v, h$  be solutions to (1.6)-(1.7) and  $E_r$  be defined as above. We prove that there are continuous functions  $C_r$  such that

$$\left| \frac{dE_r(t)}{dt} \right| \leq C_r(K, \frac{1}{\epsilon}, M, c_0, \text{vol } \mathcal{D}_t, E_{r-1}^*) E_r^*(t), \quad (1.28)$$

if  $0 \leq r \leq 4$  provided that the assumptions on  $e(h)$  hold and

$$|\theta| + \frac{1}{l_0} \leq K, \quad \text{on } \partial \mathcal{D}_t, \quad (1.29)$$

$$-\nabla_N h \geq \epsilon > 0, \quad \text{on } \partial \mathcal{D}_t, \quad (1.30)$$

$$1 \leq |\rho| \leq M, \quad \text{in } \mathcal{D}_t, \quad (1.31)$$

$$|\partial v| + |\partial h| + |\partial^2 h| + |\partial D_t h| \leq M, \quad \text{in } \mathcal{D}_t. \quad (1.32)$$

$$|D_t h| + |D_t^2 h| \leq M, \quad \text{in } \mathcal{D}_t. \quad (1.33)$$

The bounds (1.29) gives us control of geometry of the free surface  $\partial \mathcal{D}_t$ . A bound for the second fundamental form  $\theta$  gives a bound for the curvature of  $\partial \mathcal{D}_t$ , and a lower bound for the injectivity radius of the exponential map  $l_0$  measures how far off the surface is from self-intersecting. Note that for the compressible Euler's equations the bounds (1.31)-(1.33) together with the second equation of (1.6) and (1.20) imply the bounds (1.33). We only include these bounds here because we need them to hold uniformly to pass to the incompressible limit. It follows from (1.28) that the energies  $E_r(t)$  are bounded as long as the apriori  $L^\infty$  bounds above hold. On the other hand it follows from the energy bounds if  $r \geq 4$  and  $n \leq 3$  that the a priori  $L^\infty$  bounds hold up to some small positive time  $t \leq T$  (depending only on the initial energy and  $L^\infty$  bounds) if slightly stronger bounds hold initially (Proposition 5.5).

The above energy bounds remain valid uniformly as the sound speed goes to infinity (Theorem 6.2 and Proposition 6.1). The sound speed  $\kappa$  is defined by viewing  $\{p_\kappa(\rho)\}$  as a family parametrized by  $\kappa \in \mathbb{R}^+$ , such that for each  $\kappa$  we have

$$p'_\kappa(\rho)|_{\rho=1} = \kappa.$$

In this setting, we consider the Euler equations

$$\begin{cases} D_t v_\kappa = -\partial h_\kappa & \text{in } \mathcal{D}, \\ \text{div } v_\kappa = -D_t e_\kappa(h_\kappa) & \text{in } \mathcal{D}. \end{cases} \quad (1.34)$$

Here, we further assume that  $e_\kappa(h)$  satisfies

1.  $e_\kappa(h) \rightarrow 0$  as  $\kappa \rightarrow \infty$ .
2.  $|e_\kappa^{(k)}(h)| \leq c_0$ , where  $c_0$  is a fixed constant.
3.  $|e_\kappa^{(k)}(h)| \leq c_0 \sqrt{e'_\kappa(h)}$ .

The uniform (with respect to the sound speed) a priori bounds are due to that our estimates do not depend on the lower bound of  $e_\kappa^{(k)}(h)$ , which goes to 0 as  $\kappa \rightarrow \infty$ . In addition, apart from the coefficient in front of the highest order time derivative our energy does not depend in crucial

way on  $\kappa$  but uniformly (as  $\kappa \rightarrow \infty$ ) control the corresponding norms of all but the highest order time derivative. This leads to that the a priori  $L^\infty$  bounds also hold uniformly and the norms are bounded uniformly up to a fixed time. The convergence of solutions for the compressible Euler equations to the solution for the incompressible equations follows (Theorem 6.3). Specifically, we prove that  $v_\kappa, h_\kappa$  converge in  $C^2([0, T] \times \Omega)$  (in the Lagrangian coordinates defined in section 2) to a solution of the equations (1.6)-(1.7) with  $e(h) = 0$ , i.e. the incompressible Euler's equations.

Finally, in section 7 we show that for every incompressible data there are data for the compressible equations (depending on  $\kappa$ ) satisfying the required number of compatibility conditions and converging in our energy norm to the incompressible data as  $\kappa \rightarrow \infty$ .

**Remark.** For simplicity we shall only prove (1.28) for  $r \leq 4$ , which will be sufficient to get back the apriori bounds and close the argument for  $n = 2, 3$ . Our method can however be used to prove the energy bound for all order  $r$ . Since existence in  $H^N$  for some large  $N$  was shown in [13] using Nash-Moser iteration, one should expect that the solution  $v_\kappa, h_\kappa$  exist in some fixed time interval  $[0, T]$  as long as the energy bounds of order  $r \geq N$  hold.

**Remark.** One could try to define  $E_r$  with  $\widehat{D}_t := \sqrt{e'_\kappa(h)} D_t$  in place of  $D_t$ , as an analog to Disconzi-Ebin [5] and Ebin [6]. However, the resulting energies are too weak to control the evolution of

$$\mathcal{E}(t) := |(\nabla_N h(t, \cdot))^{-1}|_{L^\infty(\partial\Omega)}.$$

This is due to the a priori assumption  $|\partial D_t h| \leq M$  has to be replaced by  $|\partial \widehat{D}_t h| \leq M$ , which is weaker since  $e'_\kappa(h) \rightarrow 0$ . Although our energies are stronger we show that for every incompressible data there are data for the compressible equations converging in our energy norm.

**Remark.** We can alternatively use the modified energies defined as

$$\widehat{E}_r = \sum_{s+k=r, k \leq r-2} E_{s,k} + K_r + W_{r+1}^2, \quad (1.35)$$

which reduces the number of time derivatives involved in  $E_r$ . The statement of the main theorem still holds with  $E_r$  replaced by  $\widehat{E}_r$ .

### 1.5 Outline of the proof of the higher order energy estimate (1.28)

We conclude the introduction by showing how the time derivative of the interior terms of the energy to leading order cancel each other after integrating by parts modulo a boundary term that in turn is to leading order canceled by the time derivative of the boundary term. Let  $s + k = r$ , we have

$$\begin{aligned} \frac{d}{dt} E_{s,k} &= \int_{\mathcal{D}_t} \rho \delta^{ij} Q(\partial^s D_t^k v_i, D_t \partial^s D_t^k v_j) dx + \int_{\mathcal{D}_t} \rho e'(h) Q(\partial^s D_t^k h, D_t \partial^s D_t^k h) dx \\ &\quad + \int_{\partial \mathcal{D}_t} \rho Q(\partial^s D_t^k h, D_t \partial^s D_t^k h) \nu dS + \dots, \end{aligned} \quad (1.36)$$

where the dots stand for lower order terms. Using the commutator  $[D_t, \partial_i] = -(\partial_i v^j) \partial_j$  and the equation  $D_t v_i = -\partial_i h$  we get

$$D_t \partial^s D_t^k v_j = -\partial^s D_t^k \partial_j h + \dots = -\partial_j \partial^s D_t^k h + \dots, \quad (1.37)$$

$$D_t \partial^r h + (\partial_j h) \partial^r v^j = \partial^r D_t h + \dots, \quad (1.38)$$

$$D_t \partial^s D_t^k h = \partial^s D_t^{k+1} h + \dots \quad (1.39)$$

Hence,

$$\begin{aligned} \frac{d}{dt}E_{k,s} = & - \int_{\mathcal{D}_t} \rho(\delta^{ij} Q(\partial^s D_t^k v_i, \partial_j \partial^s D_t^k h) dx + \int_{\mathcal{D}_t} \rho e'(h) Q(\partial^s D_t^k h, \partial^s D_t^{k+1} h) dx \\ & + \int_{\partial \mathcal{D}_t} \rho Q(\partial^s D_t^k h, D_t \partial^s D_t^k h) \nu dS + \dots \end{aligned} \quad (1.40)$$

Now, if we integrate by part in the first term, we get

$$\begin{aligned} \frac{d}{dt}E_{k,s} = & \int_{\mathcal{D}_t} \rho \delta^{ij} Q(\partial_j \partial^s D_t^k v_i, \partial^s D_t^k h) dx + \int_{\mathcal{D}_t} \rho e'(h) Q(\partial^s D_t^k h, \partial^s D_t^{k+1} h) dx \\ & + \int_{\partial \mathcal{D}_t} \rho Q(\partial^s D_t^k h, D_t \partial^s D_t^k h - \nu^{-1} N_i \partial^s D_t^k v^i) \nu dS + \dots \end{aligned} \quad (1.41)$$

The terms in the first line cancel each other (up to lower-order terms) since  $\delta^{ij} \partial_j \partial^s D_t^k v_i = \partial^s D_t^k \delta^{ij} \partial_j v_i + \dots$  and  $\operatorname{div} v = -e'(h) D_t h$ .

Because our total energy of order  $r$  contain estimates of more time derivatives than space derivatives the most problematic case in which we need to estimate the boundary term above is when  $s = r$  and  $k = 0$ . Using (1.38) we see hence see that we are left with

$$\frac{d}{dt}E_{r,0} = \int_{\partial \mathcal{D}_t} \rho Q(\partial^r h, \partial^r D_t h - \partial_i h \partial^r v^i - \nu^{-1} N_i \partial^r v^i) \nu dS + \dots \quad (1.42)$$

We have choose  $\nu$  to exactly cancel the leading order term at the boundary in this case. Since  $-\nu^{-1} N_i = \partial_i h$ , the first term on the second line is inner product of  $\|\Pi \partial^r h\|_{L^2(\partial \mathcal{D}_t)}$  and plus the sum of the inner products of  $\|\Pi \partial^r D_t h\|_{L^2(\partial \mathcal{D}_t)}$ , which due to (1.24)-(1.25) we are able to control.

The proof of the energy estimate for Euler's equations outlined above is given in section 5. The proof of the energy estimate for the wave equation in section 4 and the elliptic bounds in section 3.

## 2 Lagrangian coordinate, covariant differentiation and metric, regularity of the boundary

Let us first introduce Lagrangian coordinate, under which the boundary becomes fixed. Let  $\Omega$  be the unit ball in  $\mathbb{R}^n$ , and let  $f_0 : \Omega \rightarrow \mathcal{D}_0$  to be a diffeomorphism. The Lagrangian coordinate  $(t, y)$  where  $x = x(t, y) = f_t(y)$  are given by solving

$$\frac{dx}{dt} = v(t, x(t, y)), \quad x(0, y) = f_0(y), \quad y \in \Omega. \quad (2.1)$$

The boundary becomes fixed in the new coordinate, and we introduce the notation

$$D_t = \frac{\partial}{\partial t} \Big|_{y=\text{constant}} = \frac{\partial}{\partial t} \Big|_{x=\text{constant}} + v^k \frac{\partial}{\partial x^k}. \quad (2.2)$$

to be the material derivative and

$$\partial_i = \frac{\partial}{\partial x^i} = \frac{\partial y^a}{\partial x^i} \frac{\partial}{\partial y^a}.$$

Due to (2.2), we shall also call  $D_t$  as the time derivative as well by slightly abuse of terminology.

Sometimes it is convenient to work in the Eulerian coordinate  $(t, x)$ , and sometimes it is easier



to work in the Lagrangian coordinate  $(t, y)$ . In the Lagrangian coordinate the partial derivative  $\partial_t = D_t$  has more direct significance than it in the Eulerian frame. However, this is not true for spatial derivatives  $\partial_i$ . The notion of space derivative that plays a more significant role in the Lagrangian coordinate is that the covariant differentiation with respect to the metric  $g_{ab}(t, y) = \delta_{ij} \frac{\partial x^i}{\partial y^a} \frac{\partial x^j}{\partial y^b}$ . We shall not involve covariant derivatives in our energy; instead, we use the regular Eulerian spatial derivatives. We will work mostly in the Lagrangian coordinate in this paper. However, our statements are coordinate independent.

The Euclidean metric  $\delta_{ij}$  in  $\mathcal{D}_t$  induces a metric

$$g_{ab}(t, y) = \delta_{ij} \frac{\partial x^i}{\partial y^a} \frac{\partial x^j}{\partial y^b}, \quad (2.3)$$

in  $\Omega$  for each fixed  $t$ . We will denote covariant differentiation in the  $y_a$ -coordinate by  $\nabla_a$ ,  $a = 1, \dots, n$ , and the differentiation in the  $x_i$ -coordinate by  $\partial_i$ ,  $i = 1, \dots, n$ . Here, we use the convention that differentiation with respect to Eulerian coordinates is denoted by letters  $i, j, k, l$  and with respect to Lagrangian coordinate is denoted by  $a, b, c, d$ .

The regularity of the boundary is measured by the regularity of the normal, let  $N^a$  to be the unit normal to  $\partial\Omega$ ,

$$g_{ab} N^a N^b = 1,$$

and let  $N_a = g_{ab} N^b$  denote the unit co-normal,  $g^{ab} N_a N_b = 1$ . The induced metric  $\gamma$  on the tangent space to the boundary  $T(\partial\Omega)$  extended to be 0 on the orthogonal complement in  $T(\Omega)$  is given by

$$\gamma_{ab} = g_{ab} - N_a N_b, \quad \gamma^{ab} = g^{ac} g^{bd} \gamma_{cd} = g^{ab} - N^a N^b.$$

The orthogonal projection of an  $(0, r)$  tensor  $S$  to the boundary is given by

$$(\Pi S)_{a_1, \dots, a_r} = \gamma_{a_1}^{b_1} \cdots \gamma_{a_r}^{b_r} S_{b_1, \dots, b_r},$$

where  $\gamma_a^b = g^{bc} \gamma_{ac} = \delta_a^b - N_a N^b$ . In particular, the covariant differentiation on the boundary  $\bar{\nabla}$  is given by

$$\bar{\nabla} S = \Pi \nabla S.$$

We note that  $\bar{\nabla}$  is invariantly defined since the projection and  $\nabla$  are. The second fundamental form of the boundary  $\theta$  is given by  $\theta_{ab} = (\bar{\nabla} N)_{ab}$ , and the mean curvature of the boundary  $\sigma = \text{tr} \theta = g^{ab} \theta_{ab}$ .

It is now important to compute time derivative of the metric  $D_t g$ , the normal  $D_t N$ , as well as the time derivative of corresponding measures.

**Lemma 2.1.** Let  $x = f_t(y) = x(t, y)$  be the change of variable given by

$$\frac{dx}{dt} = v(t, x(t, y)), \quad x(0, y) = f_0(y), \quad y \in \Omega,$$

and

$$g_{ab}(t, y) = \delta_{ij} \frac{\partial x^i}{\partial y^a} \frac{\partial x^j}{\partial y^b},$$

to be the induced metric. In addition, we let  $\gamma_{ab} = g_{ab} - N_a N_b$ , where  $N_a = g_{ab} N^b$  is the co-normal to  $\partial\Omega$ . Set

$$v_a(t, y) = v_i(t, x) \frac{\partial x^i}{\partial y^a}, \quad u^a = g^{ab} u_b, \quad (2.4)$$

$$d\mu_g, \quad \text{volume element with respect to the metric } g, \quad (2.5)$$

$$d\mu_\gamma, \quad \text{surface element with respect to the metric } \gamma. \quad (2.6)$$

Then

$$D_t g_{ab} = \nabla_a v_b + \nabla_b v_a, \quad (2.7)$$

$$D_t g^{ab} = -g^{ac} g^{bd} D_t g_{cd}, \quad (2.8)$$

$$D_t N_a = -\frac{1}{2} N_a (D_t g^{cd}) N_c N_d, \quad (2.9)$$

$$D_t d\mu_g = \operatorname{div} v \, d\mu_g, \quad (2.10)$$

$$D_t d\mu_\gamma = (\sigma v \cdot N) d\mu_\gamma. \quad (2.11)$$

*Proof.* We have, since the commutator  $[D_t, \frac{\partial}{\partial y}] = 0$ , and  $D_t x(t, y) = v(t, y)$ ,

$$D_t \frac{\partial x^i}{\partial y^a} = \frac{\partial v_i}{\partial y^a} = \frac{\partial x^k}{\partial y^a} \frac{\partial v_i}{\partial x^k},$$

and so

$$D_t g_{ab} = \sum_i D_t \left( \frac{\partial x^i}{\partial y^a} \frac{\partial x^i}{\partial y^b} \right) = \frac{\partial x^k}{\partial y^a} \frac{\partial v_i}{\partial x^k} \frac{\partial x^i}{\partial y^b} + \frac{\partial x^i}{\partial y^a} \frac{\partial x^k}{\partial y^b} \frac{\partial v_i}{\partial x^k} = \nabla_a v_b + \nabla_b v_a.$$

(2.8) follows from  $0 = D_t(g^{ab} g_{bc}) = D_t(g^{ab}) g_{bc} + g^{ab} D_t g_{bc}$ . (2.10) follows since in local coordinate we have  $d\mu_g = \sqrt{\det g} \, dy$  and  $D_t \det g = \det g g^{ab} D_t g_{ab} = 2 \det g \operatorname{div} v$ .

To prove (2.9), we choose the local foliation  $u$  so that  $\partial\Omega = \{y : u(y) = 0\}$  and  $u < 0$  in  $\Omega$ , then

$$N_a = \frac{\partial_a u}{\sqrt{g^{cd} \partial_c u \partial_d u}},$$

and (2.9) follows from a direct computation. Now since

$$d\mu_\gamma = \frac{\sqrt{\det g}}{\sqrt{\sum N_n^2}} dS(y),$$

where  $dS(y)$  is the Euclidean surface measure. By (2.9) we have

$$D_t d\mu_\gamma = \operatorname{div} v + \frac{1}{2} (D_t g^{cd}) N_c N_d.$$

But since  $\operatorname{div} v = g^{ab} D_t g_{ab} / 2$  and then (2.7) and (2.8) imply

$$D_t d\mu_\gamma = \frac{1}{2} g^{ab} D_t g_{ab} - \frac{1}{2} (D_t g_{ab}) N^a N^b = \gamma^{ab} \nabla_a v_b.$$

But since  $\gamma^{ab} \nabla_a v_b = \gamma^{ab} \bar{\nabla}_a (N_b v \cdot N) + \gamma^{ab} \bar{\nabla}_a \bar{v}_b$ , and  $\gamma^{ab} \bar{\nabla}_a \bar{v}_b = \operatorname{div} v|_{\partial\Omega} = 0$ , (2.11) follows.  $\square$

### 3 Estimates on a bounded domain with a moving boundary

Most of the results in this section will be stated in a coordinate-independent fashion. Throughout this section,  $\nabla$  will refer to covariant derivative with respect to the metric  $g_{ij}$  in  $\Omega$ , and  $\bar{\nabla}$  will refer to covariant differentiation on  $\partial\Omega$  with respect to the induced metric  $\gamma_{ij} = g_{ij} - N_i N_j$ . Hence, in this section (and only),  $\Omega$  will be used to denote a general domain with smooth boundary. In addition, we shall assume the normal  $N$  to  $\partial\Omega$  is extended to a vector field in the interior of  $\Omega$  satisfying  $g_{ij} N^i N^j \leq 1$  by the same way introduced in Lemma A.1.

### 3.1 Elliptic estimates

**Definition 3.1.** Let  $u : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a smooth vector field, and  $\beta_k = \beta_{Ik} = \nabla_I^r u_k$  be the  $(0, r)$ -tensor defined based on  $u_k$ , where  $\nabla_I^r = \nabla_{i_1} \cdots \nabla_{i_r}$  and  $I = (i_1, \dots, i_r)$  is the set of indices. Let  $\operatorname{div} \beta_k = \nabla_i \beta^i_k = \nabla^r \operatorname{div} u$  and  $\operatorname{curl} \beta = \nabla_i \beta_j - \nabla_j \beta_i = \nabla^r \operatorname{curl} u_{ij}$ .

**Definition 3.2.** (norms) If  $|I| = |J| = r$ , let  $g^{IJ} = g^{i_1 j_1} \cdots g^{i_r j_r}$  and  $\gamma^{IJ} = \gamma^{i_1 j_1} \cdots \gamma^{i_r j_r}$ . If  $\alpha, \beta$  are  $(0, r)$  tensors, let  $\langle \alpha, \beta \rangle = g^{IJ} \alpha_I \beta_J$  and  $|\alpha| = \langle \alpha, \alpha \rangle$ . If  $(\Pi \beta)_I = \gamma_I^J \beta_J$  is the projection, then  $\langle \Pi \alpha, \Pi \beta \rangle = \gamma^{IJ} \alpha_I \beta_J$ . Let

$$\begin{aligned} \|\beta\|_{L^2(\Omega)} &= \left( \int_{\Omega} |\beta|^2 d\mu_g \right)^{\frac{1}{2}}, \\ \|\beta\|_{L^2(\partial\Omega)} &= \left( \int_{\partial\Omega} |\beta|^2 d\mu_{\gamma} \right)^{\frac{1}{2}}, \\ \|\Pi\beta\|_{L^2(\partial\Omega)} &= \left( \int_{\partial\Omega} |\Pi\beta|^2 d\mu_{\gamma} \right)^{\frac{1}{2}}. \end{aligned}$$

We now state the following Hodge-type decomposition theorem, which serves as a main ingredient for proving the elliptic estimates.

**Theorem 3.1.** (Hodge-decomposition) Let  $\beta$  be defined in Definition 3.1. If  $|\theta| + |\frac{1}{l_0}| \leq K$ , where  $\theta$  is the second fundamental form and  $l_0$  is the injective radius defined in (1.14), then

$$|\nabla \beta|^2 \lesssim g^{ij} \gamma^{kl} \gamma^{IJ} \nabla_k \beta_{Ii} \nabla_l \beta_{Jj} + |\operatorname{div} \beta|^2 + |\operatorname{curl} \beta|^2 \quad (3.1)$$

$$\int_{\Omega} |\nabla \beta|^2 d\mu_g \lesssim \int_{\Omega} (N^i N^j g^{kl} \gamma^{IJ} \nabla_k \beta_{Ii} \nabla_l \beta_{Jj} + |\operatorname{div} \beta|^2 + |\operatorname{curl} \beta|^2 + K^2 |\beta|^2) d\mu_g. \quad (3.2)$$

*Proof.* See [1] □

**Lemma 3.2.** (Poincaré type inequalities) Let  $q : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}$  be a smooth and  $q|_{\partial\Omega} = 0$ , then

$$\|q\|_{L^2(\Omega)} \lesssim (\operatorname{vol} \Omega)^{\frac{1}{n}} \|\nabla q\|_{L^2(\Omega)}, \quad (3.3)$$

$$\|\nabla q\| \lesssim (\operatorname{vol} \Omega)^{\frac{1}{n}} \|\Delta q\|_{L^2(\Omega)}. \quad (3.4)$$

*Proof.* The first inequality is Faber-Krahns theorem, whose proof can be found in [14]. The second inequality follows from the first and integration by parts. □

**Proposition 3.3.** (Elliptic estimates) Let  $q : \Omega \rightarrow \mathbb{R}$  be a smooth function. Suppose that  $|\theta| + |\frac{1}{l_0}| \leq K$ , then we have, for any  $r \geq 2$  and  $\delta > 0$ ,

$$\|\nabla^r q\|_{L^2(\Omega)} + \|\nabla^r q\|_{L^2(\partial\Omega)} \lesssim_{K, \operatorname{vol} \Omega} \sum_{s \leq r} \|\Pi \nabla^s q\|_{L^2(\partial\Omega)} + \sum_{s \leq r-1} \|\nabla^s \Delta q\|_{L^2(\Omega)}, \quad (3.5)$$

$$\|\nabla^r q\|_{L^2(\Omega)} + \|\nabla^{r-1} q\|_{L^2(\partial\Omega)} \lesssim_{K, \operatorname{vol} \Omega} \delta \sum_{s \leq r} \|\Pi \nabla^s q\|_{L^2(\partial\Omega)} + \delta^{-1} \sum_{s \leq r-2} \|\nabla^s \Delta q\|_{L^2(\Omega)}. \quad (3.6)$$

where we have applied the convention that  $A \lesssim_{p,q} B$  means  $A \leq C_{p,q} B$ .

*Proof.* See [1] (Proposition 5.8). □

### 3.2 Estimate for the projection of a tensor to the tangent space of the boundary

The use of the projection of the tensor  $\Pi \nabla^s D_t^k h$  in the boundary part of energy (1.16) is essential to compensate the potential loss of regularity. A simple observation that will help us is that if  $q = 0$  on  $\partial\Omega$ , then  $\Pi \nabla^2 q$  contains only first-order derivative of  $q$  and all components of the second fundamental form. To be more precise, we have

$$\Pi \nabla^2 q = \bar{\nabla}^2 q + \theta \nabla_N q, \quad (3.7)$$

where the tangential component  $\bar{\nabla}^2 q = 0$  on the boundary. Furthermore, in  $L^2$  norms, (3.7) yields,

$$\|\Pi \nabla^2 q\|_{L^2(\partial\Omega)} \leq \|\theta\|_{L^\infty(\partial\Omega)} \|\nabla_N q\|_{L^2(\partial\Omega)}. \quad (3.8)$$

To prove (3.7), we first recall the components of the projection operator  $\gamma_i^j = \delta_i^j - N_i N^j$ , hence

$$\gamma_j^k \nabla_i \gamma_k^l = -\gamma_j^k \nabla_i (N_k N^l) = -\gamma_j^k \theta_{ik} N^l - \gamma_j^k N_k \theta_i^l = -\theta_{ij} N^l,$$

and so

$$\begin{aligned} \bar{\nabla}_i \bar{\nabla}_j q &= \gamma_i^{i'} \gamma_j^{j'} \nabla_{i'} \gamma_{j'}^{j''} \nabla_{j''} q \\ &= \gamma_i^{i'} \gamma_j^{j'} \gamma_{j'}^{j''} \nabla_{i'} \nabla_{j''} q + \gamma_i^{i'} \gamma_j^{j'} (\nabla_{i'} \gamma_{j'}^{j''}) \nabla_{j''} q \\ &= \gamma_i^{i'} \gamma_j^{j'} \nabla_{i'} \nabla_{j'} q - \theta_{ij} \nabla_N q. \end{aligned}$$

In general, the higher order projection formula is of the form

$$\Pi \nabla^r q = (\bar{\nabla}^{r-2} \theta) \nabla_N q + O(\nabla^{r-1} q) + O(\bar{\nabla}^{r-3} \theta),$$

which suggests the following generalization of (3.8), its detailed proof can be found in [1].

**Proposition 3.4.** (Tensor estimate) Suppose that  $|\theta| + |\frac{1}{\theta_0}| \leq K$ , and for  $q = 0$  on  $\partial\Omega$ , then for  $m = 0, 1$

$$\begin{aligned} \|\Pi \nabla^r q\|_{L^2(\partial\Omega)} &\lesssim_K \|(\bar{\nabla}^{r-2} \theta) \nabla_N q\|_{L^2(\partial\Omega)} + \sum_{l=1}^{r-1} \|\nabla^{r-l} q\|_{L^2(\partial\Omega)}, \\ &+ (\|\theta\|_{L^\infty(\partial\Omega)} + \sum_{0 \leq l \leq r-2-m} \|\bar{\nabla}^l \theta\|_{L^2(\partial\Omega)}) \left( \sum_{0 \leq l \leq r-2+m} \|\nabla^l q\|_{L^2(\partial\Omega)} \right), \quad (3.9) \end{aligned}$$

where the second line drops for  $0 \leq r \leq 4$ .

*Proof.* See [1] (Proposition 5.9). □

### 3.3 Estimate for the second fundamental form

The estimate of the second fundamental form is a direct consequence of Proposition 3.4 with  $q = h$  together with the Taylor sign condition, e.g.,  $|\nabla_N h| \geq \epsilon > 0$ .

**Proposition 3.5.** ( $\theta$  estimate) <sup>1</sup>Assume that  $0 \leq r \leq 4$ . Suppose that  $|\theta| + |\frac{1}{\theta_0}| \leq K$ , and the Taylor sign condition  $|\nabla_N h| \geq \epsilon > 0$  holds, then

$$\|\bar{\nabla}^{r-2} \theta\|_{L^2(\partial\Omega)} \lesssim_{K, \frac{1}{\epsilon}} \|\Pi \nabla^r h\|_{L^2(\partial\Omega)} + \sum_{s=1}^{r-1} \|\nabla^{r-s} h\|_{L^2(\partial\Omega)}. \quad (3.10)$$

In fact, (3.10) can both be applied to the cases when  $r > 4$  by modifying the lower order terms. We refer [1] for the details.

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<sup>1</sup>The  $\theta$  estimate suggests that the boundary regularity is in fact controlled by the boundary  $L^2$ -norm of  $h$ , with a loss of 2 derivatives.

## 4 Energy estimates for the wave equation

In this section we study the estimates for the enthalpy  $h$ . The commutator between  $D_t$  and  $\partial$  is of the form

$$[D_t, \partial_i] = -(\partial_i v^k) \partial_k. \quad (4.1)$$

If we take divergence on the first equation of (1.6), together with the fact that  $\operatorname{div} v = -D_t e(h)$  and (4.1), we obtain

$$D_t^2 e(h) - \Delta h = (\partial_i v^j)(\partial_j v^i), \quad \text{in } [0, T] \times \Omega, \quad (4.2)$$

with initial and boundary conditions

$$h|_{t=0} = h_0, \quad D_t h|_{t=0} = h_1, \quad (4.3)$$

and

$$h|_{\partial\Omega} = 0. \quad (4.4)$$

Here,  $\Delta h = \delta^{ij} \partial_i \partial_j h = \frac{1}{\sqrt{|\det g|}} \partial_a (\sqrt{|\det g|} g^{ab} \partial_b h)$ .

### 4.1 Some commutators

We are able to obtain a higher order version of (4.2) by commuting more time derivatives to it. But since our  $D_t$  no longer commutes with the spatial derivatives, we need to compute the following commutators first:

1.  $[\partial_i, D_t^k] = \sum_{l=0}^{k-1} D_t^l [\partial_i, D_t] D_t^{k-l-1}$
2.  $[\Delta, D_t] = \Delta v^j \partial_j + 2\partial^i v^j \partial_i \partial_j = -\partial^j (D_t e(h)) \partial_j + \partial^j \operatorname{curl}_{kj} v + 2\partial^i v^j \partial_i \partial_j$ , where  $\partial^i = \delta^{ik} \partial_k$

The second equality is because  $\Delta v_j = \sum_k \partial_k \partial_k v_j = \partial_j \operatorname{div} v + \sum_k \partial_k \operatorname{curl}_{kj} v$

3.  $[\Delta, D_t^{r-1}] = \sum_{l=0}^{r-2} D_t^l [\Delta, D_t] D_t^{r-l-2}$

Although  $D_t$  and  $\partial$  are not commutative, (4.1) implies that the commutator between  $D_t$  and  $\partial$  is free from time derivative. In general,  $[D_t^k, \partial]$  is a product of mixed space-time derivative where each component depends on at most  $k-1$  time derivatives. In fact for  $k \leq 4$ , each component of  $[\partial, D_t^k]$  and  $[\Delta, D_t^k]$  can be controlled by the a priori assumptions. This can be seen by the simplified version of the commutators, by expressing them in the format of main terms + lower order terms. To do it, we would like to introduce the following short-hand notations first.

**Definition 4.1.** (Symmetric dot product) Let  $[D_t, \partial] = -(\partial v) \tilde{\cdot} \partial$ , where the symmetric dot product  $(\partial v) \tilde{\cdot} \partial$  is define component-wise by  $((\partial v) \tilde{\cdot} \partial)_i = \partial_i v^k \partial_k$ . In general, we have

$$[D_t, \partial^r] = \sum_{s=0}^{r-1} \partial^s [D_t, \partial] \partial^{r-s-1} = \sum_{s=0}^{r-1} - \binom{r}{s+1} (\partial^{1+s} v) \tilde{\cdot} \partial^{r-s}, \quad (4.5)$$

where

$$((\partial^{1+s} v) \tilde{\cdot} \partial^{r-s})_{i_1, \dots, i_r} = \frac{1}{r!} \sum_{\sigma \in S_r} (\partial_{i_{\sigma_1} \dots i_{\sigma_{1+s}}}^{1+s} v^k) (\partial_{k, i_{\sigma_{s+2}} \dots i_{\sigma_r}}^s),$$

where  $S_r$  is the  $r$ -symmetric group.

Now, the commutators  $[\partial, D_t^k], k \geq 2$  and  $[\Delta, D_t^{r-1}], r \geq 3$  can be rewritten as

$$[\partial, D_t^k] = \sum_{l_1+l_2=k-1} c_{l_1, l_2} (\partial D_t^{l_1} v) \cdot (\partial D_t^{l_2} v) + \sum_{l_1+\dots+l_n=k-n+1, n \geq 3} d_{l_1, \dots, l_n} (\partial D_t^{l_1} v) \cdots (\partial D_t^{l_{n-1}} v) (\partial D_t^{l_n} v), \quad (4.6)$$

and

$$\begin{aligned} [\Delta, D_t^{r-1}] &= \sum_{l_1+l_2=r-2} c_{l_1, l_2} (\Delta D_t^{l_1} v) \cdot (\partial D_t^{l_2} v) + \sum_{l_1+l_2=r-2} c_{l_1, l_2} (\partial D_t^{l_1} v) \cdot (\partial^2 D_t^{l_2} v) \\ &\quad + \sum_{l_1+\dots+l_n=r-n, n \geq 3} d_{l_1, \dots, l_n} (\partial D_t^{l_1} v) \cdots (\partial D_t^{l_{n-1}} v) \cdot (\Delta D_t^{l_n} v) \cdot (\partial D_t^{l_1} v) \\ &\quad + \sum_{l_1+\dots+l_n=r-n, n \geq 3} e_{l_1, \dots, l_n} (\partial D_t^{l_1} v) \cdots (\partial D_t^{l_{n-1}} v) \cdot (\partial^2 D_t^{l_n} v) \cdot (\partial D_t^{l_1} v) \\ &\quad + \sum_{l_1+\dots+l_n=r-n, n \geq 3} f_{l_1, \dots, l_n} (\partial D_t^{l_1} v) \cdots (\partial D_t^{l_{n-1}} v) \cdot (\partial D_t^{l_n} v) \cdot (\partial^2 D_t^{l_1} v), \end{aligned} \quad (4.7)$$

where the regular dot product is defined by the trace of the symmetric dot.

## 4.2 The Energies $W_r(t)$

By commuting  $D_t^{r-1}$  on both sides of (4.2), we obtain the higher order wave equation

$$e'(h) D_t^{r+1} h - \Delta D_t^{r-1} h = f_r + g_r, \quad (4.8)$$

where

$$f_r = D_t^{r-1} (\partial v \cdot \partial v) + [D_t^{r-1}, \Delta] h, \quad (4.9)$$

and  $g_r$  is sum of terms of the form

$$e^m(h) (D_t^{i_1} h) \cdots (D_t^{i_m} h), \quad i_1 + \cdots + i_m = r+1, \quad 1 \leq i_1 \leq \cdots \leq i_m \leq r. \quad (4.10)$$

Now, let us define the energy

$$W_r(t) = \frac{1}{2} \|\sqrt{e'(h)} D_t^r h\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\nabla D_t^{r-1} h\|_{L^2(\Omega)}^2. \quad (4.11)$$

The following estimate holds for  $W_r$ :

**Theorem 4.1.** (Energy estimates for  $W_r$ ) Let  $W_r$  be defined as (4.11), then

$$\frac{dW_r^2}{dt} \lesssim_M W_r^2 + W_r (\|f_r\|_{L^2(\Omega)} + \|g_r\|_{L^2(\Omega)}). \quad (4.12)$$

*Proof.* We start by differentiating  $\|\sqrt{e'(h)} D_t^r h\|_{L^2(\Omega)}^2$ , and since  $|e^{(r)}(h)| \leq c_0$ ,  $|D_t h| \leq M$ , and  $D_t J = (\operatorname{div} v) J$ , where  $J = \sqrt{\det g}$ , one has

$$\begin{aligned} \frac{d}{dt} \frac{1}{2} \int_{\Omega} e'(h) D_t^r h \cdot D_t^r h J dy &\lesssim \int_{\Omega} e'(h) D_t^{r+1} h \cdot D_t^r h J dy + W_r^2 \\ &= \int_{\Omega} \Delta D_t^{r-1} h \cdot D_t^r h J dy + \int_{\Omega} (f_r + g_r) D_t^r h J dy + W_r^2 \\ &= - \int_{\Omega} \nabla D_t^{r-1} h \cdot \nabla D_t^r h J dy + \int_{\Omega} (f_r + g_r) D_t^r h J dy + W_r^2. \end{aligned} \quad (4.13)$$

Now by (4.1),  $\nabla D_t^r h = D_t \nabla D_t^{r-1} h + \nabla v \cdot \nabla D_t^{r-1} h$ , and so that

$$-\int_{\Omega} \nabla D_t^{r-1} h \cdot \nabla D_t^r h J dy = -\frac{1}{2} \frac{d}{dt} \|\nabla D_t^{r-1} h\|_{L^2(\Omega)}^2 + \int_{\Omega} \nabla D_t^{r-1} h (\nabla v \cdot \nabla D_t^{r-1} h) J dy, \quad (4.14)$$

where the last term on the right hand side of (4.14) is included in  $W_r$  since  $|\nabla v| \leq M$ .  $\square$

### 4.3 Estimates for $\|f_r\|_{L^2(\Omega)}$

Let us now analyse  $\|f_r\|$  and  $\|g_r\|$ . By adopting our notations used in (4.6)-(4.7), we are able to express  $f_r$  as

$$\begin{aligned} f_r = & \sum_{l_1+l_2=r-1} c_{l_1, l_2} (\nabla D_t^{l_1} v) \cdot (\nabla D_t^{l_2} v) + \sum_{l_1+l_2=r-2} d_{l_1, l_2} (\Delta D_t^{l_1} v) \cdot (\nabla D_t^{l_2} h) \\ & + \sum_{l_1+l_2=r-2} e_{l_1, l_2} (\nabla D_t^{l_1} v) \cdot (\nabla^2 D_t^{l_2} h) + \text{error terms}, \end{aligned} \quad (4.15)$$

where the "error terms" refer to the terms generated by the commutators, which are of the form

$$\begin{aligned} e_r = & \sum_{l_1+\dots+l_n=r+1-n, n \geq 3} g_{l_1, \dots, l_n} (\partial D_t^{l_3} v) \cdots (\partial D_t^{l_n} v) \cdot (\partial D_t^{l_1} v) \cdot (\partial D_t^{l_2} v) \\ & + \sum_{l_1+\dots+l_n=r-n, n \geq 3} e_{l_1, \dots, l_n} (\partial D_t^{l_3} v) \cdots (\partial D_t^{l_n} v) \cdot (\partial^2 D_t^{l_1} v) \cdot (\partial D_t^{l_2} h) \\ & + \sum_{l_1+\dots+l_n=r-n, n \geq 3} f_{l_1, \dots, l_n} (\partial D_t^{l_3} v) \cdots (\partial D_t^{l_n} v) \cdot (\partial D_t^{l_1} v) \cdot (\partial^2 D_t^{l_2} h). \end{aligned} \quad (4.16)$$

We need to estimate  $\|f_r\|_{L^2(\Omega)}$  for  $r = 2, 3, 4, 5$ . Since our estimates include mixed space-time derivatives, we would like to use the following more appealing notations.

**Definition 4.2.** (Mixed Sobolev norms) let  $u(t, \cdot) : \mathbb{R}^n \rightarrow \mathbb{R}$  be a smooth function. We define

$$\begin{aligned} \|u\|_{r, 0} &= \sum_{s+k=r, k < r} \|\nabla^s D_t^k u\|_{L^2(\Omega)}, \\ \|u\|_r &= \|u\|_{r, 0} + \|\sqrt{e'(h)} D_t^r h\|_{L^2(\Omega)}. \end{aligned}$$

It is worth mentioning that when  $r \leq 5$ , the error terms of  $f_r$  generated by commuting  $D_t$  and the spatial derivatives can be controlled linearly by mixed Sobolev norms of  $v$  and  $h$  of order at most  $r-1$ . On the other hand, in order to estimate the time derivative of  $W_{r+1}$  in the next section, we have to make sure that the  $r$ -th order Sobolev norms in our estimates for  $\|f_r\|_{L^2(\Omega)}$ ,  $3 \leq r \leq 5$  do not include  $\|\nabla^r h\|_{L^2(\Omega)}$  and  $\|v\|_r$ .

#### 4.3.1 When $r=2$

The main terms involved in  $f_2$  can be bounded by

$$\begin{aligned} \|(\nabla D_t v)(\nabla v)\|_{L^2(\Omega)} &\leq \|\nabla v\|_{L^\infty} \|\nabla^2 h\|_{L^2(\Omega)}, \\ \|(\Delta v)(\nabla h)\|_{L^2(\Omega)} &\leq \|\nabla h\|_{L^\infty} \|\nabla^2 v\|_{L^2(\Omega)}, \\ \|(\nabla v)(\nabla^2 h)\|_{L^2(\Omega)} &\leq \|\nabla v\|_{L^\infty} \|\nabla^2 h\|_{L^2(\Omega)}. \end{aligned}$$

Since the error terms in  $f_2$  is of the form  $\nabla v \cdot \nabla v \cdot \nabla v$ , we get

$$\|f_2\|_{L^2(\Omega)} \lesssim_M \|\nabla^2 v\|_{L^2(\Omega)} + \|\nabla^2 h\|_{L^2(\Omega)} + \|\nabla v\|_{L^2(\Omega)}.$$

### 4.3.2 When $r=3$

The first and the third terms of  $f_3$  can be bounded by

$$\begin{aligned} \|(\nabla D_t^2 v)(\nabla v)\|_{L^2(\Omega)} + \|(\nabla D_t v)(\nabla D_t v)\|_{L^2(\Omega)} &\lesssim_M \|\nabla^2 D_t h\|_{L^2(\Omega)} + \|\nabla^2 v\|_{L^2(\Omega)} + \|\nabla^2 h\|_{L^2(\Omega)}, \\ \|(\nabla v)(\nabla^2 h)\|_{L^2(\Omega)} + \|(\nabla D_t v)(\nabla^2 h)\|_{L^2(\Omega)} &\lesssim_M \|\nabla^2 h\|_{L^2(\Omega)}, \end{aligned}$$

respectively. To bound the second term, it is easy to see that by the wave equation (4.8) and the fact that  $|e^{(r)}(h)| \leq c_0$ , we get

$$\|(\Delta D_t v)(\nabla h)\|_{L^2(\Omega)} = \|(\nabla \Delta h)(\nabla h)\|_{L^2(\Omega)} \lesssim_M \|\nabla D_t^2 h\|_{L^2(\Omega)} + \sum_{j=1,2} \|\nabla^j v\|_{L^2(\Omega)} + \|h\|_{2,0},$$

and<sup>2</sup>

$$\|(\Delta v)(\nabla D_t h)\|_{L^2(\Omega)} \leq \|\nabla D_t h\|_{L^\infty} \|\nabla^2 v\|_{L^2(\Omega)} \lesssim_M \|\nabla^2 v\|_{L^2(\Omega)}.$$

The higher order terms in  $e_3$  are essentially bounded by the corresponding terms in  $f_r$ , for  $r \leq 3$ , we just estimated times  $|\nabla v|_{L^\infty}$ , apart from a term of the form  $\nabla v \cdot \nabla^2 v \cdot \nabla h$  which can be estimated by  $\|\nabla^2 v\|_{L^2}$ . Hence,

$$\|f_3\|_{L^2(\Omega)} \lesssim_M \|\nabla^2 D_t h\|_{L^2(\Omega)} + \|\nabla D_t^2 h\|_{L^2(\Omega)} + \|h\|_{2,0} + \sum_{j=1,2} \|\nabla^j v\|_{L^2(\Omega)}.$$

### 4.3.3 When $r=4$

The first term of  $f_4$  can be bounded by

$$\begin{aligned} \sum_{l_1+l_2=3} \|c_{l_1,l_2}(\nabla D_t^{l_1} v)(\nabla D_t^{l_2} v)\|_{L^2(\Omega)} &\leq \|\nabla v\|_{L^\infty} \|\nabla D_t^3 v\|_{L^2(\Omega)} + \|\nabla^2 h\|_{L^\infty} \|\nabla D_t^2 v\|_{L^2(\mathcal{D}_t)} \\ &\lesssim_M \|\nabla^2 D_t^2 h\|_{L^2(\Omega)} + \|\nabla^2 D_t h\|_{L^2(\Omega)} + \|\nabla[D_t^2, \nabla]h\|_{L^2(\Omega)} + \|\nabla(\nabla v \cdot \nabla h)\|_{L^2(\Omega)} \\ &\lesssim_M \|\nabla^2 D_t^2 h\|_{L^2(\Omega)} + \|\nabla^2 D_t h\|_{L^2(\Omega)} + \sum_{j=2,3} \|h\|_{j,0} + \|\nabla^2 v\|_{L^2(\Omega)}. \end{aligned}$$

Whereas the third term can be bounded by

$$\sum_{l_1+l_2=2} \|e_{l_1,l_2}(\nabla D_t^{l_1} v)(\nabla^2 D_t^{l_2} h)\|_{L^2(\Omega)} \lesssim_M \|\nabla D_t^2 v\|_{L^2(\Omega)} + \|\nabla^2 D_t h\|_{L^2(\Omega)} + \|\nabla^2 D_t^2 h\|_{L^2(\Omega)}.$$

To bound the second term, by interpolation (A.5) we have

$$\begin{aligned} \sum_{l_1+l_2=2} \|(\Delta D_t^{l_1} v) \cdot (\nabla D_t^{l_2} h)\|_{L^2(\Omega)} &\lesssim_{K,M} \|\nabla v\|_{L^\infty} \sum_{j=1,2} \|\nabla^j D_t^2 h\|_{L^2(\Omega)} + \|D_t^2 h\|_{L^\infty} \sum_{j=2,3} \|\nabla^j v\|_{L^2(\Omega)} \\ &\quad + \|\nabla^3 D_t h\|_{L^2(\Omega)} + \sum_{j=2,3} (\|\nabla^j v\|_{L^2(\Omega)} + \|\nabla^j h\|_{L^2(\Omega)}) \\ &\lesssim_{K,M} \|\nabla^2 D_t^2 h\|_{L^2(\Omega)} + \|\nabla^3 D_t h\|_{L^2(\Omega)} + \sum_{j=2,3} (\|h\|_{j,0} + \|\nabla^j v\|_{L^2(\Omega)}). \quad (4.17) \end{aligned}$$

<sup>2</sup>We must include  $\|\nabla D_t h\|_{L^\infty}$  in our a priori assumptions. Since otherwise we would have to estimate  $\|(\Delta v)(\nabla D_t h)\|_{L^2(\Omega)}$  by interpolation, which contributes to  $\|\nabla^3 v\|_{L^2(\Omega)}$  and it is part of  $E_3$ . Hence we would lose one derivative when estimating the second order boundary  $L^2$  norms.



Most of the terms in  $e_4$  can be bounded by corresponding terms in  $f_r$ , for  $r \leq 4$ , and similar terms in  $e_3$  times a priori assumptions, apart from terms of the form  $\nabla v \cdot \nabla^2 D_t v \cdot \nabla h$ , whose  $L^2$  norm can be bounded by  $\|\nabla^3 h\|_{L^2(\Omega)}$ .

Therefore, we sum up and get

$$\|f_4\|_{L^2(\Omega)} \lesssim_{K,M} \|\nabla^3 D_t h\|_{L^2(\Omega)} + \|\nabla^2 D_t^2 h\|_{L^2(\Omega)} + \sum_{j=2,3} \|h\|_{j,0} + \sum_{j=1,2,3} \|\nabla^j v\|_{L^2(\Omega)}.$$

#### 4.3.4 When $r = 5$ and $n \leq 4$

The first and the third terms of  $f_5$  can be estimated by through similar method as above.

$$\begin{aligned} \sum_{l_1+l_2=4} \|(\nabla D_t^{l_1} v)(\nabla D_t^{l_2} v)\|_{L^2(\Omega)} + \sum_{l_1+l_2=3} \|(\nabla D_t^{l_1} v)(\nabla^2 D_t^{l_2} h)\|_{L^2(\Omega)} \\ \lesssim_{K,M} \|\nabla^2 D_t^3 h\|_{L^2(\Omega)} + \sum_{1 \leq i \leq 4} \|v\|_{i,0} + \sum_{2 \leq i \leq 4} \|h\|_{i,0}. \end{aligned}$$

We need the Sobolev lemma (A.3) to bound  $\sum_{l_1+l_2=3} d_{l_1,l_2} \|(\Delta D_t^{l_1} v)(\nabla D_t^{l_2} h)\|_{L^2(\Omega)}$ , whose terms are bounded by

$$\|\Delta v \cdot \nabla D_t^3 h\|_{L^2(\Omega)} \lesssim_K \left( \sum_{j=2,3} \|\nabla^j v\|_{L^2(\Omega)} \right) \left( \sum_{j=1,2} \|\nabla^j D_t^3 h\|_{L^2(\Omega)} \right), \quad (4.18)$$

$$\|\Delta D_t v \cdot \nabla D_t^2 h\|_{L^2(\Omega)} \lesssim_K |\nabla^2 h|_{L^\infty} \sum_{j=1,2} \|\nabla^j D_t^2 h\|_{L^2(\Omega)} + |D_t^2 h|_{L^\infty} \sum_{j=3,4} \|\nabla^j h\|_{L^2(\Omega)},$$

$$\|\Delta D_t^2 v \cdot \nabla D_t h\|_{L^2(\Omega)} \lesssim_M \|\nabla^2 D_t^2 v\|_{L^2(\Omega)},$$

and

$$\begin{aligned} \|\Delta D_t^3 v \cdot \nabla h\|_{L^2(\Omega)} \lesssim_M \|\nabla \Delta D_t^2 h\|_{L^2(\Omega)} + \|\Delta [D_t^2, \nabla] h\|_{L^2(\Omega)} \\ \lesssim_{K,M} \|\nabla^3 D_t^2 h\|_{L^2(\Omega)} + \sum_{j=2,3} \|v\|_{j,0} + \sum_{j=3,4} \|h\|_{j,0}, \end{aligned}$$

respectively. Most of the terms in the error term  $e_5$  are essentially bounded by corresponding terms in  $f_r$ , for  $r \leq 5$ , and similar terms in  $e_3$  and  $e_4$  times a priori assumptions, apart from the terms of the form  $\nabla v \cdot \nabla^2 D_t^2 v \cdot \nabla h$ , which is estimated by  $\|\nabla^2 D_t^2 v\|_{L^2(\Omega)}$ .

Hence<sup>3</sup>

$$\begin{aligned} \|f_5\|_{L^2(\Omega)} \lesssim_{K,M} \|\nabla^3 D_t^2 h\|_{L^2(\Omega)} + \left( \sum_{j=2,3} \|\nabla^j v\|_{L^2(\Omega)} \right) \|\nabla^2 D_t^3 h\|_{L^2(\Omega)} \\ + \sum_{1 \leq i \leq 4} \|v\|_{i,0} + \left( \sum_{j=2,3} \|\nabla^j v\|_{L^2(\Omega)} \right) \|\nabla D_t^3 h\|_{L^2(\Omega)} + \sum_{2 \leq i \leq 4} \|h\|_{i,0}. \end{aligned}$$

<sup>3</sup> It can be seen that our estimates started to loss linearity in the highest orders when  $r \geq 5$ .

#### 4.4 Estimates for $\|g_r\|_{L^2(\Omega)}$ for $r \leq 4$ .

Since  $g_r$  is a sum of  $e^m(h)(D_t^{i_1}h) \cdots (D_t^{i_m}h)$ ,  $i_1 + \cdots + i_m = r + 1$  and  $1 \leq i_1 \leq \cdots \leq i_m \leq r$ , one cannot apply interpolation inequalities on estimating  $g_r$ . But since terms of the form  $\|(D_t h)^l D_t^k h\|_{L^2(\Omega)}$  and  $\|(D_t^2 h)^l D_t^k h\|_{L^2(\Omega)}$  can be bounded by  $\|D_t^k h\|_{L^2(\Omega)}$  times  $|D_t h|_{L^\infty}$  or  $|D_t^2 h|_{L^\infty}$ . Further, since  $|e^{(m)}(h)| \leq c_0 \sqrt{e'(h)}$ , we get for  $r \leq 4$  that

$$\|g_r\|_{L^2(\Omega)} \lesssim_{M, c_0} \sum_{i \leq r} \|\sqrt{e'(h)} D_t^i h\|_{L^2(\Omega)}. \quad (4.19)$$

#### 4.5 Estimates for $\|g_r\|_{L^2(\Omega)}$ for $r = 5$ and $n \leq 4$ .

The only difference for estimating  $g_5$  is that it contains a quadratic term  $e''(h)D_t^3 h \cdot D_t^3 h$ , whose  $L^2$  norm is bounded via Sobolev lemma (A.3). Hence,

$$\|e''(h)(D_t^3 h)^2\| \leq |D_t^3 h|_{L^\infty} \|\sqrt{e'(h)} D_t^3 h\|_{L^2(\Omega)},$$

but

$$|D_t^3 h|_{L^\infty} \lesssim_{K, \text{vol } \Omega} \|\nabla^2 D_t^3 h\|_{L^2(\Omega)} + \|\nabla D_t^3 h\|_{L^2(\Omega)},$$

where we have used the fact  $\|D_t^3 h\|_{L^2(\Omega)} \lesssim_{\text{vol } \Omega} \|\nabla D_t^3 h\|_{L^2(\Omega)}$  as a consequence of (3.3). Therefore, we conclude

$$\|g_5\|_{L^2(\Omega)} \lesssim_{K, M, c_0, \text{vol } \Omega} \sum_{j \leq 5} \|\sqrt{e'(h)} D_t^j h\|_{L^2(\Omega)} + (\|\nabla^2 D_t^3 h\|_{L^2(\Omega)} + \|\nabla D_t^3 h\|_{L^2(\Omega)}) \|\sqrt{e'(h)} D_t^3 h\|_{L^2(\Omega)}. \quad (4.20)$$

## 5 Energy estimates for the Euler equations

We are now ready to prove our main theorem.

**Proposition 5.1.** Let  $E_r$  be defined as (1.16), then there are continuous functions  $C_r$  such that, for  $t \in [0, T]$  and  $r \leq 4$ ,

$$\left| \frac{dE_r(t)}{dt} \right| \leq C_r(K, \frac{1}{\epsilon}, M, c_0, \text{vol } \mathcal{D}_t, E_{r-1}^*) E_r^*(t), \quad (5.1)$$

provided that the assumptions on  $e(h)$  and the a priori bounds (1.29)-(1.33) hold.

Our estimates will mostly be in the Lagrangian coordinates, but we shall compute the time derivative  $\frac{d}{dt} E_r$  in Eulerian coordinate, since then we do not need to worry about the Christoffel symbols.

### 5.1 Computing $\frac{d}{dt} E_r$

We first compute  $\frac{d}{dt} E_{s,k}(t)$ , where  $E_{s,k}$  is defined as (1.17), when  $s > 0$ .

$$\begin{aligned} \frac{d}{dt} E_{s,k} &= \frac{1}{2} \int_{\mathcal{D}_t} \rho D_t(\delta^{ij} Q(\partial^s D_t^k v_i, \partial^s D_t^k v_j)) dx + \frac{1}{2} \int_{\mathcal{D}_t} \rho D_t(e'(h) Q(\partial^s D_t^k h, \partial^s D_t^k h)) dx \\ &+ \frac{1}{2} \int_{\partial \mathcal{D}_t} \rho D_t(Q(\partial^s D_t^k h, \partial^s D_t^k h) \nu) - Q(\partial^s D_t^k h, \partial^s D_t^k h) \nu (\sigma v \cdot N) + \rho Q(\partial^s D_t^k h, \partial^s D_t^k h) D_t \nu dS. \end{aligned} \quad (5.2)$$

The estimates (A.1)-(A.4) together with a priori assumptions imply<sup>4</sup>

$$|D_t q^{ij}| \lesssim M, \quad |\partial q^{ij}| \lesssim M + K, \quad |\sigma v \cdot N|_{L^\infty(\partial\Omega)} \lesssim K + M,$$

$$|D_t \nu|_{L^\infty(\partial\Omega)} = |D_t(-\nabla_N h)^{-1}|_{L^\infty(\partial\Omega)} \lesssim 1 + \frac{1}{M},$$

and

$$D_t \gamma^{ij} = -2\gamma^{im}\gamma^{jn}(\frac{1}{2}D_t g_{mn}). \quad (5.3)$$

Since  $|D_t q^{ij}| \lesssim M$  in the interior and on the boundary  $q^{ij} = \gamma^{ij}$ , and by (5.3)  $D_t \gamma$  is tangential, so that (5.2) can then be reduced to

$$\begin{aligned} \frac{d}{dt} E_{s,k} &\leq \int_{\mathcal{D}_t} \rho \delta^{ij} Q(D_t \partial^s D_t^k v_i, \partial^s D_t^k v_j) dx + \int_{\mathcal{D}_t} \rho e'(h) Q(D_t \partial^s D_t^k h, \partial^s D_t^k h) dx \\ &\quad + \int_{\partial\mathcal{D}_t} \rho Q(D_t \partial^s D_t^k h, \partial^s D_t^k h) \nu dS + C(K, L, M)(E_{s,k} + \|h\|_{r,0}^2 + \|v\|_{r,0}^2). \end{aligned} \quad (5.4)$$

Now, our commutators (4.5) and (4.6) yield, since  $D_t v_i = -\partial_i h$ ,

$$D_t \partial^s D_t^k v_i = -\partial^s D_t^k \partial_i h + \sum_{0 \leq m \leq s-1} c_{msk} (\partial^{m+1} v) \cdot \partial^{s-m} D_t^k v_i, \quad (5.5)$$

$$D_t \partial^r h + (\partial_j h) \partial^r v^j = \partial^r D_t h + \sum_{0 \leq m \leq r-2} d_{mr} (\partial^{m+1} v) \cdot \partial^{r-m} h, \quad (5.6)$$

$$D_t \partial^s D_t^k h = \partial^s D_t^{k+1} h + \sum_{0 \leq m \leq s-1} d_{msk} (\partial^{m+1} v) \cdot \partial^{s-m} D_t^k h, \quad \text{for } k \geq 1. \quad (5.7)$$

We control the term  $\|(\partial^{m+1} v) \cdot \partial^{s-m} D_t^k v_i\|_{L^2(\mathcal{D}_t)}$  in (5.5) and  $\|(\partial^{m+1} v) \cdot \partial^{s-m} D_t^k h\|_{L^2(\mathcal{D}_t)}$  in (5.7) for  $s+k=r$ ,  $k < r$  and  $r \leq 4$ .

- Since  $r \leq 4$  and  $k < r$  imply  $k \leq 3$ , the term  $\|(\partial^{m+1} v) \cdot \partial^{s-m} D_t^k h\|_{L^2(\mathcal{D}_t)}$  can be bounded by

$$|D_t^k h|_{L^\infty} \sum_{j \leq s+1} \|\partial^j v\|_{L^2(\mathcal{D}_t)} + |\partial v|_{L^\infty} \sum_{j \leq s} \|\partial^j D_t^k h\|_{L^2(\Omega)},$$

when  $k=1, 2$ .

For  $k=3$ , we have

$$\|(\partial v) \cdot \partial D_t^3 h\|_{L^2(\mathcal{D}_t)} \lesssim_M \|\partial D_t^3 h\|_{L^2(\mathcal{D}_t)},$$

and for  $k=0$ ,

$$\|(\partial^{m+1} v) \cdot \partial^{r-m} h\|_{L^2(\mathcal{D}_t)} \lesssim_K |\partial v|_{L^\infty} \sum_{j \leq r} \|\partial^j h\|_{L^2(\mathcal{D}_t)} + |\partial h|_{L^\infty} \sum_{j \leq r} \|\partial^j v\|_{L^2(\mathcal{D}_t)}.$$

- The term  $\|(\partial^{m+1} v) \cdot \partial^{s-m} D_t^k v_i\|_{L^2(\mathcal{D}_t)}$  with  $k \geq 1$  can be re-expressed as

$$\|(\partial^{m+1} v) \cdot \partial^{s-m+1} D_t^{k-1} h\|_{L^2(\mathcal{D}_t)} + \|(\partial^{m+1} v) \cdot \partial^{s-m} [D_t^{k-1}, \partial] h\|_{L^2(\mathcal{D}_t)},$$

since  $k-1 \leq 2$ , the second term is bounded by  $\sum_{i \leq r-1} (\|v\|_{i,0} + \|h\|_{i,0})$  and the first can be bounded similarly as above.

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<sup>4</sup>We refer Section 5 of [1] for the detailed proof

The above analysis shows that the  $L^2$  norm of the sum in (5.5)-(5.7) contribute only to  $\|v\|_{r,0}$  and  $\|h\|_{r,0}$  with  $r < 4$ . Hence,

$$\begin{aligned} \frac{d}{dt} E_{s,k} \leq & - \int_{\mathcal{D}_t} \rho(\delta^{ij} Q(\partial^s D_t^k v_i, \partial^s D_t^k \partial_j h)) dx + \int_{\mathcal{D}_t} \rho e'(h) Q(\partial^s D_t^k h, \partial^s D_t^{k+1} h) dx \\ & + \int_{\partial \mathcal{D}_t} \rho Q(\partial^s D_t^k h, D_t \partial^s D_t^k h) \nu dS + C(K, M)(\|v\|_{r,0} + \|h\|_{r,0}) \left( \sum_{i \leq r} \|v\|_{i,0} + \|h\|_{i,0} \right). \end{aligned} \quad (5.8)$$

Now (4.6) yields,

$$\begin{aligned} \|\partial^s D_t^k \partial h - \partial^{s+1} D_t^k h\|_{L^2(\mathcal{D}_t)} \lesssim & \sum_{l_1+l_2=k-1} \|\partial^s (\partial D_t^{l_1} v \tilde{\partial} D_t^{l_2} h)\|_{L^2(\mathcal{D}_t)} \\ & + \sum_{l_1+\dots+l_n=k-n+1, n \geq 3} \|\partial^s (\partial D_t^{l_1} v \dots \partial D_t^{l_{n-1}} v \tilde{\partial} D_t^{l_n} h)\|_{L^2(\mathcal{D}_t)}, \end{aligned} \quad (5.9)$$

and the last two terms are bounded by  $\sum_{i \leq r} (\|h\|_{i,0} + \|v\|_{i,0})$ . Therefore,

$$\begin{aligned} \frac{d}{dt} E_{s,k} \leq & - \int_{\mathcal{D}_t} \rho(\delta^{ij} Q(\partial^s D_t^k v_i, \partial_j \partial^s D_t^k h)) dx + \int_{\mathcal{D}_t} \rho e'(h) Q(\partial^s D_t^k h, \partial^s D_t^{k+1} h) dx \\ & + \int_{\partial \mathcal{D}_t} \rho Q(\partial^s D_t^k h, D_t \partial^s D_t^k h) \nu dS + C(K, M)(\|v\|_{r,0} + \|h\|_{r,0}) \left( \sum_{i \leq r} \|v\|_{i,0} + \|h\|_{i,0} \right). \end{aligned} \quad (5.10)$$

If we integrate by parts in the first term

$$\begin{aligned} & \int_{\mathcal{D}_t} \rho \delta^{ij} Q(\partial^s D_t^k \partial_i v_j, \partial^s D_t^k h) dx + \int_{\mathcal{D}_t} \rho e'(h) Q(\partial^s D_t^k h, \partial^s D_t^{k+1} h) dx \\ & + \int_{\partial \mathcal{D}_t} \rho Q(\partial^s D_t^k h, D_t \partial^s D_t^k h - \nu^{-1} N_i \partial^s D_t^k v^i) \nu dS + C(K, M)(\|v\|_{r,0} + \|h\|_{r,0}) \left( \sum_{i \leq r} \|v\|_{i,0} + \|h\|_{i,0} \right). \end{aligned} \quad (5.11)$$

But since  $\partial^s D_t^{k+1} e(h)$  equals  $e'(h) \partial^s D_t^{k+1} h$  plus a sum of terms of the form

$$e^{(m)}(h) (\partial^{i_1} D_t^{j_1} h) \dots (\partial^{i_m} D_t^{j_m} h),$$

where

$$(i_1 + j_1) + \dots + (i_m + j_m) \leq r + 1, \quad 1 \leq i_1 + j_1 \leq \dots \leq i_m + j_m \leq r.$$

Therefore,

$$\begin{aligned} \int_{\mathcal{D}_t} \rho \delta^{ij} Q(\partial^s D_t^k \partial_i v_j, \partial^s D_t^k h) dx &= \int_{\mathcal{D}_t} \rho Q(\partial^s D_t^k \operatorname{div} v, \partial^s D_t^k h) dx \\ &= - \int_{\mathcal{D}_t} \rho Q(\partial^s D_t^{k+1} e(h), \partial^s D_t^k h) dx \\ &\leq - \int_{\mathcal{D}_t} \rho e'(h) Q(\partial^s D_t^{k+1} h, \partial^s D_t^k h) dx + C(K, M) \left( \sum_{i \leq r} \|h\|_{i,0} \right)^2, \end{aligned} \quad (5.12)$$

so the first integral in (5.11) cancels with the second term.

We recall  $\nu = -(\partial_N h)^{-1}$ , so that  $\nu^{-1}N_i = \partial_i h$ . Hence, the boundary term in (5.11) becomes

$$\sum_{k+s=r, s>0} \int_{\partial\mathcal{D}_t} \rho Q(\partial^s D_t^k h, D_t \partial^s D_t^k h + (\partial_i h)(\partial^s D_t^k v^i) \nu) dS. \quad (5.13)$$

Now, since (5.6) and (5.7), (5.13) becomes sum of the boundary inner product of  $\Pi \partial^s D_t^k h$  and

$$\Pi(D_t \partial^r h + (\partial_j h) \partial^r v^j) = \Pi \partial^r D_t h + \sum_{0 \leq m \leq r-2} d_{mr} \Pi((\partial^{m+1} v) \tilde{\cdot} \partial^{r-m} h), \quad (5.14)$$

$$\Pi(D_t \partial^s D_t^k h + (\partial_i h)(\partial^s D_t^k v^i)) = \Pi \partial^s D_t^{k+1} h + \Pi(\partial_i h)(\partial^s D_t^k v^i) + \sum_{0 \leq m \leq s-1} d_{mr} \Pi((\partial^{m+1} v) \tilde{\cdot} \partial^{s-m} D_t^k h), \quad (5.15)$$

for  $k = 0$  and  $k > 0$ , respectively.

In addition, when  $s = 0$ ,

$$\frac{d}{dt} E_{0,r} \leq - \int_{\mathcal{D}_t} \rho \delta^{ij} (D_t^r \partial_i h) (D_t^r v_j) dx + \int_{\mathcal{D}_t} \rho e'(h) (D_t^{r+1} h) (D_t^r h) dx + C(M) \|\sqrt{e'(h)} D_t^r h\|_{L^2(\mathcal{D}_t)}^2, \quad (5.16)$$

where we have used the fact that  $e''(h) \leq c_0 \sqrt{e'(h)}$ . Furthermore, since

$$\begin{aligned} \|\partial D_t^r h - \partial D_t^r h\|_{L^2(\mathcal{D}_t)} &\lesssim \sum_{l_1+l_2=r-1} \|\partial D_t^{l_1} v \tilde{\cdot} \partial D_t^{l_2} h\|_{L^2(\mathcal{D}_t)} \\ &\quad + \sum_{l_1+\dots+l_n=r-n+1, n \geq 3} \|\partial D_t^{l_1} v \dots \partial D_t^{l_{n-1}} v \tilde{\cdot} \partial D_t^{l_n} h\|_{L^2(\mathcal{D}_t)} \end{aligned} \quad (5.17)$$

(5.16) becomes, after integrating by parts on the first integral on the RHS of (5.16),

$$\begin{aligned} \frac{d}{dt} E_{0,r} &\leq \int_{\mathcal{D}_t} \rho \delta^{ij} (D_t^r h) (D_t^r \operatorname{div} v) dx + \int_{\mathcal{D}_t} \rho e'(h) (D_t^{r+1} h) (D_t^r h) dx \\ &\quad + C(M) \|\sqrt{e'(h)} D_t^r h\|_{L^2(\mathcal{D}_t)}^2 + C(M) \sum_{i \leq r} (\|h\|_{i,0} + \|v\|_{i,0})^2. \end{aligned} \quad (5.18)$$

But since

$$D_t^r \operatorname{div} v = -D_t^{r+1} e(h) = -e'(h) D_t^{r+1} h - g_r,$$

and because  $\|\sqrt{e'(h)} D_t^r h\|_{L^2(\mathcal{D}_t)}$  is part of  $\|h\|_r$ , (5.18) becomes

$$\frac{d}{dt} E_{0,r} \leq C(M) \sum_{i \leq r} (\|h\|_i + \|v\|_{i,0})^2. \quad (5.19)$$

Furthermore, let  $K_r$  be defined as (1.18), we have

$$\frac{d}{dt} K_r = 2 \int_{\mathcal{D}_t} \rho |\partial^{r-1} \operatorname{curl} v| \cdot |D_t \partial^{r-1} \operatorname{curl} v| dx. \quad (5.20)$$

But since the curl satisfies the equation

$$D_t \text{curl}_{ij} v = -(\partial_i v^k)(\text{curl}_{kj} v) + (\partial_j v^k)(\text{curl}_{ki} v),$$

then

$$\begin{aligned} |D_t \partial^{r-1} \text{curl} v| &\leq |\partial^{r-1} D_t \text{curl} v| + \sum_{0 \leq m \leq r-2} e_{mr} (\partial^{m+1} v) \cdot \partial^{r-1-m} \text{curl} v \\ &\lesssim \sum_{0 \leq m \leq r-1} e_{mr} (\partial^{m+1} v) \cdot \partial^{r-1-m} \text{curl} v. \end{aligned} \quad (5.21)$$

The term  $\|(\partial^{m+1} v) \cdot \partial^{r-1-m} \text{curl} v\|_{L^2(\mathcal{D}_t)}$  can be bounded by

$$|\partial v|_{L^\infty} \sum_{j \leq r-1} \|\partial^j \text{curl} v\|_{L^2(\mathcal{D}_t)} + |\text{curl} v|_{L^\infty} \sum_{j \leq r-1} \|\partial^{j+1} v\|_{L^2(\mathcal{D}_t)}. \quad (5.22)$$

On the other hand,

$$\begin{aligned} \frac{dW_{r+1}^2}{dt} &= 2W_{r+1} \frac{dW_{r+1}}{dt} \lesssim W_{r+1} (W_{r+1} + \|f_{r+1}\|_{L^2(\mathcal{D}_t)} + \|g_{r+1}\|_{L^2(\mathcal{D}_t)}) \\ &\lesssim E_r + \sqrt{E_r} (\|f_{r+1}\|_{L^2(\mathcal{D}_t)} + \|g_{r+1}\|_{L^2(\mathcal{D}_t)}). \end{aligned} \quad (5.23)$$

This comes from the energy estimates for the wave equation, e.g., Theorem 4.1.

Therefore, we have proved:

**Theorem 5.2.** Let  $E_r$  be defined as (1.16), for  $r \leq 4$  we have

$$\begin{aligned} \left| \frac{dE_r}{dt} \right| &\lesssim_{K,M} E_r + \sum_{k+s=r, k,s>0} \left( \|\Pi \partial^s D_t^k h\|_{L^2(\partial \mathcal{D}_t)} \left( \|\Pi \partial^s D_t^{k+1} h\|_{L^2(\partial \mathcal{D}_t)} \right. \right. \\ &\quad \left. \left. + \|\Pi(\partial_i h)(\partial^s D_t^k v^i)\|_{L^2(\partial \mathcal{D}_t)} + \sum_{0 \leq m \leq s-1} \|\Pi((\partial^{m+1} v) \cdot \partial^{s-m} D_t^k h)\|_{L^2(\partial \mathcal{D}_t)} \right) \right) \\ &\quad + \|\Pi \partial^r h\|_{L^2(\partial \mathcal{D}_t)} \left( \|\Pi \partial^r D_t h\|_{L^2(\partial \mathcal{D}_t)} + \sum_{0 \leq m \leq r-2} \|\Pi((\partial^{m+1} v) \cdot \partial^{r-m} h)\|_{L^2(\partial \mathcal{D}_t)} \right) \\ &\quad + C(K, M) \left( \sum_{i \leq r} \|v\|_{i,0} + \|h\|_i \right)^2 + \frac{dW_{r+1}^2}{dt}. \end{aligned} \quad (5.24)$$

**Remark.** (5.14) is essential in our energy estimates since  $\Pi \partial^r v$  is cancelled by the commutator, since there is no way to control  $\Pi \partial^r v$  on the boundary due to the loss of regularity. However, we can control  $\Pi \partial^s D_t^k v$  in (5.15) on the boundary for  $k > 0$  since it can be reduced to  $\|\Pi \partial^{s+1} D_t^{k-1} h\|_{L^2(\partial \mathcal{D}_t)}$  modulo error terms, which can then be controlled by elliptic estimates.

**Definition 5.1.** (Mixed boundary Sobolev norm) let  $u(t, \cdot) : \mathbb{R}^n \rightarrow \mathbb{R}$  be a smooth function. We define

$$\langle\langle u \rangle\rangle_r = \sum_{k+s=r} \|\nabla^s D_t^k u\|_{L^2(\partial \Omega)}.$$

Now, let us get back to Lagrangian coordinate. Based on the computation we have as well as (A.8), controlling  $\frac{d}{dt}$  requires to bound

$$\|v\|_{r,0}, \|h\|_r, \sum_{j \leq r-1} \|\nabla^j v\|_{L^2(\partial \Omega)}, \langle\langle h \rangle\rangle_r,$$

and

$$\sum_{k+s=r, s \geq 2} \|\Pi \nabla^s D_t^{k+1} h\|_{L^2(\partial\Omega)}.$$

**Theorem 5.3.** With the a priori assumptions (1.29)-(1.33) hold, we have for  $0 \leq r \leq 4$ ,

$$\|v\|_{r,0} + \|h\|_r \leq C(K, M, c_0, \text{vol } \mathcal{D}_t) \sqrt{E_r^*}. \quad (5.25)$$

In addition to that,

$$\|D_t h\|_{r+\langle\langle h \rangle\rangle_r} \leq C_r(K, M, c_0, \frac{1}{\epsilon}, \text{vol } \Omega) \sqrt{E_r^*}, \quad 0 \leq r \leq 3 \quad (5.26)$$

$$\|D_t h\|_{4+\langle\langle h \rangle\rangle_4} \leq C_r(K, M, c_0, \frac{1}{\epsilon}, \text{vol } \Omega) (1 + \sqrt{E_3^*}) \sqrt{E_4^*}. \quad (5.27)$$

We shall prove (5.25) in Section 5.2, and (5.26)-(5.27) in Section 5.3.

## 5.2 Interior estimates, bounds for $\|v\|_{r,0}, \|h\|_r$

Our strategy is to first apply Theorem 3.1 to control  $\|v\|_{r,0}$  in terms of the energies  $E_r$  and  $L^2$  norm of  $h$ , and then we will apply our elliptic estimate (3.6) to control  $\|h\|_r$ . We shall only focus on  $r = 4$ , since the other cases follow from a similar method. Now, since

$$\|v\|_{4,0} \lesssim_M \|\nabla^4 v\|_{L^2(\Omega)} + \|\nabla^4 h\|_{L^2(\Omega)} + \|\nabla^3 D_t h\|_{L^2(\Omega)} + \|\nabla^2 D_t^2 h\|_{L^2(\Omega)} + \sum_{1 \leq i \leq 3} (\|v\|_{i,0} + \|h\|_{i,0}). \quad (5.28)$$

So the terms of order 4 except for  $\|\nabla^4 v\|$  can be combined with  $\|h\|_4$ . Now, Theorem 3.1 yields,

$$\|\nabla^4 v\|_{L^2(\Omega)} \lesssim \sqrt{E_4} + \|\nabla^3 \text{div } v\|_{L^2(\Omega)}. \quad (5.29)$$

We recall that  $\text{div } v = -e'(h) D_t h$ , hence

$$\|\nabla^4 v\|_{L^2(\Omega)} \lesssim \sqrt{E_4} + \sum_{1 \leq j \leq 4} \|h\|_{j,0}. \quad (5.30)$$

To bound  $\|h\|_4$ , since (3.6) provides, for each  $k, s$  that  $k + s = r$ ,

$$\|\nabla^s D_t^k h\|_{L^2(\Omega)} \lesssim_{K,M,\text{vol } \Omega} \|\Pi \nabla^s D_t^k h\|_{L^2(\partial\Omega)} + \sum_{0 \leq j \leq s-2} \|\nabla^j \Delta D_t^k h\|_{L^2(\Omega)}, \quad (5.31)$$

for  $s \geq 2$  (and so  $k \leq 2$ ). The term  $\|\Pi \nabla^s D_t^k h\|_{L^2(\partial\Omega)}$  bounded by  $(\|\nabla h\|_{L^\infty(\partial\Omega)} E_r)^{\frac{1}{2}}$ , by the construction of  $E_r$ . Furthermore, by the wave equation (4.8)

$$\begin{aligned} \sum_{0 \leq j \leq s-2, 2 \leq s \leq 4, s+k=r} \|\nabla^j \Delta D_t^k h\|_{L^2(\Omega)} &\lesssim \sum_{0 \leq j \leq 2} \|\nabla^j D_t^2 e(h)\|_{L^2(\Omega)} \\ &+ \sum_{0 \leq j \leq 1} \|\nabla^j D_t^3 e(h)\|_{L^2(\Omega)} + \|D_t^4 e(h)\|_{L^2(\Omega)} + \sum_{1 \leq i \leq 3} (\|v\|_{i,0} + \|h\|_i). \end{aligned} \quad (5.32)$$

But since  $|e^{(l)}(h)| \leq c_0$ , by the same way as we did to control  $\|g_r\|_{L^2(\Omega)}$ ,

$$\sum_{0 \leq j \leq s-2, 2 \leq s \leq 4, k+s=r} \|\nabla^j \Delta D_t^k h\|_{L^2(\Omega)} \lesssim_{K,M,c_0,\text{vol } \Omega} \|\nabla^2 D_t^2 h\|_{L^2(\Omega)} + W_4 + \sum_{1 \leq i \leq 3} (\|v\|_{i,0} + \|h\|_i), \quad (5.33)$$

and if we apply (3.6) again with  $q = D_t^2 h$  to  $\|\nabla^2 D_t^2 h\|_{L^2(\Omega)}$ , we have

$$\begin{aligned} \|\nabla^2 D_t^2 h\|_{L^2(\Omega)} &\lesssim_{K,M,c_0,\text{vol } \Omega} \|\Pi \nabla^2 D_t^2 h\|_{L^2(\partial\Omega)} + \|\Delta D_t^2 h\|_{L^2(\Omega)} \\ &\lesssim_{K,M,c_0,\text{vol } \Omega} \sqrt{E_4^*} + W_4 + \sum_{1 \leq i \leq 3} (\|v\|_{i,0} + \|h\|_i). \end{aligned}$$

In addition, under the inductive assumption that  $\sum_{1 \leq i \leq 3} (\|v\|_{i,0} + \|h\|_i)$  is already bounded<sup>5</sup> by  $\sqrt{E_3^*} + W_3$ , and together with (5.28), we get

$$\|v\|_{4,0} + \|h\|_4 \lesssim_{K,M,c_0,\text{vol } \Omega} W_4^* + \sqrt{E_4^*}. \quad (5.34)$$

Now, if we apply similar analysis to  $\|v\|_{r,0}, \|h\|_r$  for  $r = 2, 3$  and by induction, we get that for each  $r \leq 4$ ,

$$\sum_{1 \leq i \leq r} (\|v\|_{i,0} + \|h\|_i) \lesssim_{K,M,c_0,\text{vol } \Omega} \sqrt{E_r^*} + W_r^*. \quad (5.35)$$

Now, since  $W_r^*$  is part of the energy  $\sqrt{E_{r-1}^*}$ , we have proved (5.25).

### 5.3 Boundary estimates, bounds for $\sum_{j \leq r-1} \|\nabla^j v\|_{L^2(\partial\Omega)}, \langle\langle h \rangle\rangle_r$ and $\|\bar{\nabla}^{r-2} \theta\|_{L^2(\partial\Omega)}$

The control of  $\sum_{j \leq r-1} \|\nabla^j v\|_{L^2(\partial\Omega)}$  follows directly from the estimate of  $\sum_{j \leq r} \|\nabla^j v\|_{L^2(\Omega)}$  by trace Theorem (Theorem A.9). On the other hand, we shall not estimate  $\langle\langle h \rangle\rangle_r$  alone; instead, we estimate<sup>6</sup>  $\|D_t h\|_r + \langle\langle h \rangle\rangle_r$  by (3.6). This has to be done since we need to estimate  $\|f_{r+1}\|_{L^2(\mathcal{D}_t)}$  and  $\|g_{r+1}\|_{L^2(\mathcal{D}_t)}$  by  $E_r$ . We will first do the cases when  $r = 2, 3$  in order to get a general idea.

#### 5.3.1 When $r = 2$

We estimate the mixed boundary  $L^2$  norm  $\langle\langle h \rangle\rangle_2$  by (3.5)

$$\begin{aligned} \langle\langle h \rangle\rangle_2 &\lesssim_{K,M,\text{vol } \Omega} \|\Pi \nabla^2 h\|_{L^2(\partial\Omega)} + \sum_{j \leq 1} \|\nabla^j \Delta h\|_{L^2(\Omega)} + \|\Delta D_t h\|_{L^2(\Omega)} \\ &\lesssim_{K,M,c_0,\text{vol } \Omega} \sqrt{E_2} + \sum_{j \leq 1} \|\nabla^j D_t^2 h\|_{L^2(\Omega)} + \|\sqrt{e'(h)} D_t^3 h\|_{L^2(\Omega)} + \sum_{i \leq 2} (\|v\|_{i,0} + \|h\|_{i,0}) \\ &\lesssim_{K,M,c_0,\text{vol } \Omega} \sqrt{E_2} + W_3 + \sum_{i \leq 2} (\|v\|_{i,0} + \|h\|_{i,0}), \end{aligned} \quad (5.36)$$

and by (3.6) we get, for each  $\delta > 0$  that

$$\|D_t h\|_2 \lesssim_{K,M,c_0,\text{vol } \Omega} \delta \|\Pi \nabla^2 D_t h\|_{L^2(\partial\Omega)} + \delta^{-1} \|\Delta D_t h\|_{L^2(\Omega)} + W_3. \quad (5.37)$$

Now if we combine the interior and boundary estimates, we have for  $0 < \delta < 1$  that

$$\begin{aligned} \|D_t h\|_2 + \langle\langle h \rangle\rangle_2 &\lesssim_{K,M,c_0,\text{vol } \Omega} \sqrt{E_2} + W_3 + \delta \|\Pi \nabla^2 D_t h\|_{L^2(\partial\Omega)} + \delta^{-1} \|\Delta D_t h\|_{L^2(\Omega)} + \sum_{i \leq 2} (\|v\|_{i,0} + \|h\|_{i,0}) \\ &\lesssim_{K,M,c_0,\text{vol } \Omega} \sqrt{E_2} + \delta \|\Pi \nabla^2 D_t h\|_{L^2(\partial\Omega)} + \delta^{-1} (W_3 + \sum_{i \leq 2} (\|v\|_{i,0} + \|h\|_{i,0})). \end{aligned} \quad (5.38)$$

<sup>5</sup>We mention here that the lowest order terms, namely  $\|\nabla h\|$  and  $\|D_t h\|$ , are bounded via Lemma 3.2.

<sup>6</sup>The reason that we use the norm  $\|D_t h\|_r$  instead of  $\|h\|_{r+1}$  is because the latter involves  $\|\nabla^{r+1} h\|$  which, after applying the elliptic and tensor estimates, gives  $\|(\bar{\nabla}^{r-1} \theta) \nabla_N h\|_{L^2(\partial\Omega)}$  but  $\|\bar{\nabla}^{r-1} \theta\|_{L^2(\partial\Omega)}$  can only be controlled by  $E_{r+1}$ .



Further, (5.25) would imply

$$W_3 + \sum_{i \leq 2} (\|v\|_{i,0} + \|h\|_{i,0}) \lesssim_{K,M,c_0,\text{vol } \Omega} \sqrt{E_2^*}.$$

Since by (3.9) we have

$$\delta \|\Pi \nabla^2 D_t h\|_{L^2(\partial \Omega)} \lesssim_K \delta (\|\nabla_N D_t h\|_{L^\infty(\partial \Omega)} \|\theta\|_{L^2(\partial \Omega)} + \sum_{j \leq 1} \|\nabla^j D_t h\|_{L^2(\partial \Omega)}). \quad (5.39)$$

Now if we take  $\delta = \delta(K, M, \text{vol } \Omega)$  to be sufficiently small, the last term on the RHS can be combined with  $\langle \langle h \rangle \rangle_2$  on the left (since  $D_t h = 0$  on  $\partial \Omega$ ). Since  $\|\theta\|_{L^2(\partial \Omega)} \leq \epsilon^{-1} \|\Pi \nabla^2 h\|_{L^2(\partial \Omega)}$ , and so the first term is part of  $\sqrt{E_2}$ . Therefore,

$$\|D_t h\|_2 + \langle \langle h \rangle \rangle_2 \lesssim_{K,M,c_0,\frac{1}{\epsilon},\text{vol } \Omega} \sqrt{E_2^*} + W_3 \lesssim \sqrt{E_2^*},$$

since  $W_3$  is part of  $\sqrt{E_2}$ .

### 5.3.2 When $r = 3$

By (3.5), we get

$$\begin{aligned} \langle \langle h \rangle \rangle_3 &\lesssim_{K,M,\text{vol } \Omega} \sum_{k+s=3, s>0} \|\Pi \nabla^s D_t^k h\|_{L^2(\partial \Omega)} + \sum_{j \leq 2} \|\nabla^j \Delta h\|_{L^2(\Omega)} + \sum_{j \leq 1} \|\nabla^j \Delta D_t h\|_{L^2(\Omega)} + \|\Delta D_t^2 h\|_{L^2(\Omega)} \\ &\lesssim_{K,M,c_0,\text{vol } \Omega} \sqrt{E_3} + \sum_{j \leq 2} \|\nabla^j D_t^2 h\|_{L^2(\Omega)} + \sum_{j \leq 1} \|\nabla^j D_t^3 h\|_{L^2(\Omega)} + \|\sqrt{e'(h)} D_t^4 h\|_{L^2(\Omega)} + \sum_{i \leq 3} (\|v\|_{i,0} + \|h\|_{i,0}), \end{aligned} \quad (5.40)$$

together with (5.25) we have

$$\langle \langle h \rangle \rangle_3 \lesssim_{K,M,\text{vol } \Omega} \sqrt{E_3^*} + W_4^* + \|\nabla^2 D_t^2 h\|_{L^2(\Omega)}, \quad (5.41)$$

where the last term is part of  $\|D_t h\|_3$ .

On the other hand, by (3.6) with  $0 < \delta < 1$  we get

$$\begin{aligned} \|D_t h\|_3 &\lesssim_{K,M,\text{vol } \Omega} \delta (\|\Pi \nabla^3 D_t h\|_{L^2(\partial \Omega)} + \|\Pi \nabla^2 D_t^2 h\|_{L^2(\partial \Omega)}) \\ &\quad + \delta^{-1} \left( \sum_{j \leq 1} \|\nabla^j \Delta D_t h\|_{L^2(\Omega)} + \|\Delta D_t^2 h\|_{L^2(\Omega)} \right) + W_4 \\ &\lesssim_{K,M,c_0,\text{vol } \Omega} \delta (\|\Pi \nabla^3 D_t h\|_{L^2(\partial \Omega)} + \|\Pi \nabla^2 D_t^2 h\|_{L^2(\partial \Omega)}) + \delta^{-1} (W_4 + \sum_{i \leq 3} (\|v\|_{i,0} + \|h\|_{i,0})). \end{aligned} \quad (5.42)$$

Now (3.9) implies

$$\begin{aligned} \delta (\|\Pi \nabla^3 D_t h\|_{L^2(\partial \Omega)} + \|\Pi \nabla^2 D_t^2 h\|_{L^2(\partial \Omega)}) &\lesssim_K \\ \delta (\|\bar{\nabla} \theta\|_{L^2(\partial \Omega)} \|\nabla_N D_t h\|_{L^\infty(\partial \Omega)} + \|\theta\|_{L^\infty(\partial \Omega)} \|\nabla_N D_t^2 h\|_{L^2(\partial \Omega)}) & \\ + \sum_{0 \leq j \leq 2} \|\nabla^j D_t h\|_{L^2(\partial \Omega)} + \sum_{0 \leq j \leq 1} \|\nabla^j D_t^2 h\|_{L^2(\partial \Omega)}. & \end{aligned} \quad (5.43)$$

Now let  $\delta$  to be sufficiently small, and so the last three terms on the second line can be absorbed into  $\sum_{j \leq 3} \langle \langle h \rangle \rangle_j$ . In addition, since we have just proved that

$$\|\nabla^2 h\|_{L^2(\partial\Omega)} + \|\nabla D_t h\|_{L^2(\partial\Omega)} \lesssim_{K,M,c_0,\text{vol}\Omega} \sqrt{E_2^*},$$

and so by (3.10) we have

$$\|\bar{\nabla}\theta\|_{L^2(\partial\Omega)} \lesssim_{K,\frac{1}{\epsilon}} \sqrt{E_3} + \sum_{1 \leq j \leq 2} \|\nabla^j h\|_{L^2(\partial\Omega)} \lesssim_{K,M,c_0,\frac{1}{\epsilon},\text{vol}\Omega} \sqrt{E_3^*}. \quad (5.44)$$

Therefore, if we combine the estimates for  $\langle \langle h \rangle \rangle_3$ ,  $\|D_t h\|_3$  and  $\|\bar{\nabla}\theta\|_{L^2(\partial\Omega)}$ , as well as the lower order  $L^2$  norms, we get by (5.25) that

$$\sum_{1 \leq i \leq 3} (\|v\|_{i,0} + \|h\|_i + \langle \langle h \rangle \rangle_i) + \|D_t h\|_3 \lesssim_{K,M,c_0,\frac{1}{\epsilon},\text{vol}(\Omega)} W_4^* + \sqrt{E_3^*}. \quad (5.45)$$

Therefore, since  $W_4^*$  is part of  $\sqrt{E_3^*}$ , we conclude

$$\|D_t h\|_3 + \langle \langle h \rangle \rangle_3 \lesssim_{K,M,c_0,\frac{1}{\epsilon},\text{vol}(\Omega)} \sqrt{E_3^*}.$$

### 5.3.3 When $r = 4$

The estimates for  $\langle \langle h \rangle \rangle_4$  and  $\|D_t h\|_4$  follows from the same analysis that we applied for the previous cases.

$$\begin{aligned} \langle \langle h \rangle \rangle_4 &\lesssim_{K,M,\text{vol}\Omega} \sum_{k+s=4, s>0} \|\Pi \nabla^s D_t^k h\|_{L^2(\partial\Omega)} + \sum_{j \leq 3} \|\nabla^j \Delta h\|_{L^2(\Omega)} \\ &\quad + \sum_{j \leq 2} \|\nabla^j \Delta D_t h\|_{L^2(\Omega)} + \sum_{j \leq 1} \|\nabla^j \Delta D_t^2 h\|_{L^2(\Omega)} + \|\Delta D_t^3 h\|_{L^2(\Omega)} \\ &\lesssim_{K,M,c_0,\text{vol}\Omega} \sqrt{E_4} + W_5^* + \|\nabla^2 D_t^3 h\|_{L^2(\Omega)} + \sum_{j \leq 4} (\|v\|_{i,0} + \|h\|_{i,0}), \end{aligned} \quad (5.46)$$

and for  $0 < \delta < 1$ ,

$$\begin{aligned} \|D_t h\|_4 &\lesssim_{K,M,\text{vol}\Omega} \delta (\|\Pi \nabla^4 D_t h\|_{L^2(\Omega)} + \|\Pi \nabla^3 D_t^2 h\|_{L^2(\Omega)} + \|\Pi \nabla^2 D_t^3 h\|_{L^2(\Omega)}) \\ &\quad + \delta^{-1} \left( \sum_{j \leq 2} \|\nabla^j \Delta D_t h\|_{L^2(\Omega)} + \sum_{j \leq 1} \|\nabla^j \Delta D_t^2 h\|_{L^2(\Omega)} + \|\Delta D_t^3 h\|_{L^2(\Omega)} \right) + W_5 \\ &\lesssim_{K,M,c_0,\text{vol}\Omega} \delta (\|\Pi \nabla^4 D_t h\|_{L^2(\Omega)} + \|\Pi \nabla^3 D_t^2 h\|_{L^2(\Omega)} + \|\Pi \nabla^2 D_t^3 h\|_{L^2(\Omega)}) + \delta^{-1} \left( \sum_{j \leq 4} (\|v\|_{i,0} + \|h\|_{i,0}) + W_5 \right). \end{aligned} \quad (5.47)$$

The  $L^2$  norm of the projected tensors can be estimated by

$$\begin{aligned} &\delta (\|\Pi \nabla^4 D_t h\|_{L^2(\partial\Omega)} + \|\Pi \nabla^2 D_t^3 h\|_{L^2(\partial\Omega)}) \\ &\lesssim_K \delta (\|\nabla_N D_t h\|_{L^\infty(\partial\Omega)} \|\bar{\nabla}^2 \theta\|_{L^2(\partial\Omega)} + \|\theta\|_{L^\infty(\partial\Omega)} \|\nabla_N D_t^3 h\|_{L^2(\partial\Omega)} \\ &\quad + \sum_{j \leq 3} \|\nabla^j D_t h\|_{L^2(\partial\Omega)} + \sum_{j \leq 1} \|\nabla^j D_t^3 h\|_{L^2(\partial\Omega)}), \end{aligned} \quad (5.48)$$

In addition,

$$\|\bar{\nabla}^2 \theta\|_{L^2(\Omega)} \lesssim_{K, \frac{1}{\epsilon}} \|\Pi \nabla^4 h\|_{L^2(\partial\Omega)} + \sum_{i=1}^3 \|\nabla^i h\|_{L^2(\partial\Omega)} \lesssim_{K, M, c_0, \frac{1}{\epsilon}, \text{vol } \Omega} \sqrt{E_4^*},$$

and so  $\|\Pi \nabla^4 D_t h\|_{L^2(\partial\Omega)}$  and  $\|\Pi \nabla^2 D_t^3 h\|_{L^2(\partial\Omega)}$  can be treated similarly as we did in the previous cases. On the other hand,

$$\delta \|\Pi \nabla^3 D_t^2 h\|_{L^2(\partial\Omega)} \lesssim_K \delta (\|(\nabla_N D_t^2 h) \bar{\nabla} \theta\|_{L^2(\partial\Omega)} + \sum_{j \leq 2} \|\nabla^j D_t^2 h\|_{L^2(\partial\Omega)}). \quad (5.49)$$

The first term  $\|(\nabla_N D_t^2 h) \bar{\nabla} \theta\|_{L^2(\partial\Omega)}$  is bounded via Gagliardo-Nirenberg interpolation inequality (Theorem A.7) if  $\Omega \in \mathbb{R}^3$  (e.g.,  $\partial\Omega \in \mathbb{R}^2$ ),

$$\begin{aligned} \|(\nabla_N D_t^2 h) \bar{\nabla} \theta\|_{L^2(\partial\Omega)} &\leq \|\nabla_N D_t^2 h\|_{L^4} \|\bar{\nabla} \theta\|_{L^4} \lesssim_K \|\nabla_N D_t^2 h\|_{L^2}^{\frac{1}{2}} \|\bar{\nabla} \theta\|_{L^2}^{\frac{1}{2}} \|\nabla D_t^2 h\|_{H^1(\partial\Omega)}^{\frac{1}{2}} \|\bar{\nabla} \theta\|_{H^1(\partial\Omega)}^{\frac{1}{2}} \\ &\lesssim_{K, M, \frac{1}{\epsilon}, \text{vol } \Omega} \sqrt{E_3^*} (\|\nabla D_t^2 h\|_{H^1(\partial\Omega)} + \|\bar{\nabla} \theta\|_{H^1(\partial\Omega)}) \lesssim_{K, M, c_0, \frac{1}{\epsilon}, \text{vol } \Omega} \sqrt{E_3^*} \sqrt{E_4^*} + \sqrt{E_3^*} \|\nabla D_t^2 h\|_{H^1(\partial\Omega)}, \end{aligned} \quad (5.50)$$

where the last term  $\|\nabla D_t^2 h\|_{H^1(\partial\Omega)}$  is part of  $\langle\langle h \rangle\rangle_4$ .

If  $\Omega \in \mathbb{R}^2$ , we have

$$\begin{aligned} \|(\nabla_N D_t^2 h) \bar{\nabla} \theta\|_{L^2(\partial\Omega)} &\leq \|\nabla_N D_t^2 h\|_{L^\infty(\partial\Omega)} \|\bar{\nabla} \theta\|_{L^2(\partial\Omega)} \lesssim_K \left( \sum_{j \leq 2} \|\nabla^j D_t^2 h\|_{L^2(\partial\Omega)} \right) \|\bar{\nabla} \theta\|_{L^2(\partial\Omega)} \\ &\lesssim_{K, M, c_0, \frac{1}{\epsilon}, \text{vol } \Omega} \sqrt{E_3^*} \left( \sum_{j \leq 2} \|\nabla^j D_t^2 h\|_{L^2(\partial\Omega)} \right). \end{aligned} \quad (5.51)$$

Now, if we combine the estimates for  $\langle\langle h \rangle\rangle_4$ ,  $\|D_t h\|_4$  and  $\|\bar{\nabla}^2 \theta\|_{L^2(\partial\Omega)}$ , as well as the lower order  $L^2$  norms, we get

$$\begin{aligned} \sum_{1 \leq i \leq 4} (\|v\|_{i,0} + \|h\|_i + \langle\langle h \rangle\rangle_i) + \|D_t h\|_4 &\lesssim_{K, M, c_0, \frac{1}{\epsilon}, \text{vol } \Omega} \delta^{-1} \sqrt{E_4^*} + \delta \sqrt{E_3^*} \sqrt{E_4^*} + \delta (\|\theta\|_{L^\infty(\partial\Omega)} \|\nabla_N D_t^3 h\|_{L^2(\partial\Omega)} \\ &+ \sum_{j \leq 3} \|\nabla^j D_t h\|_{L^2(\partial\Omega)} + \sum_{j \leq 2} \|\nabla^j D_t^2 h\|_{L^2(\partial\Omega)} + \sum_{j \leq 1} \|\nabla^j D_t^3 h\|_{L^2(\partial\Omega)} + \sqrt{E_3^*} \|\nabla D_t^2 h\|_{H^1(\partial\Omega)}). \end{aligned} \quad (5.52)$$

Therefore, with  $\delta$  chosen to be of the form

$$\frac{C(K, M, c_0, \frac{1}{\epsilon}, \text{vol } \Omega)}{2(1 + \sqrt{E_3^*})},$$

where  $C$  is a continuous function that is sufficiently small, the above inequality implies

$$\sum_{1 \leq i \leq 4} (\|v\|_{i,0} + \|h\|_i + \langle\langle h \rangle\rangle_i) + \|D_t h\|_4 \lesssim_{K, M, c_0, \frac{1}{\epsilon}, \text{vol } \Omega} (1 + \sqrt{E_3^*}) \sqrt{E_4^*}, \quad (5.53)$$

and so

$$\|D_t h\|_4 + \langle\langle h \rangle\rangle_4 \lesssim_{K, M, c_0, \frac{1}{\epsilon}, \text{vol } \Omega} (1 + \sqrt{E_3^*}) \sqrt{E_4^*}. \quad (5.54)$$

#### 5.4 Bounds for $\sum_{k+s=r, s \geq 2} \|\Pi \nabla^s D_t^{k+1} h\|_{L^2(\partial\Omega)}$

By (3.9), since  $D_t h = 0$ , we have

$$\sum_{k+s=r, s \geq 2} \|\Pi \nabla^s D_t^{k+1} h\|_{L^2(\partial\Omega)} \lesssim_K \sum_{k+s=r, s \geq 2} \|\bar{\nabla}^{s-2} \theta (\nabla_N D_t^{k+1} h)\|_{L^2(\partial\Omega)} \quad (5.55)$$

$$+ \sum_{k+s=r, s \geq 2} \sum_{j=1}^{s-1} \|\nabla^{s-j} D_t^{k+1} h\|_{L^2(\partial\Omega)}, \quad (5.56)$$

where the first term is bounded similarly by the arguments we had in the previous section. The second term is part of  $\langle\langle h \rangle\rangle_r$ .

#### 5.5 Bounds for $\frac{dW_{r+1}^2}{dt}$

We recall that we have

$$\frac{dW_{r+1}^2}{dt} \lesssim E_r + \sqrt{E_r} (\|f_{r+1}\|_{L^2(\Omega)} + \|g_{r+1}\|_{L^2(\Omega)}). \quad (5.57)$$

Therefore, it suffices to bound  $\|f_{r+1}\|_{L^2(\Omega)}$  and  $\|g_{r+1}\|_{L^2(\Omega)}$  by  $\sqrt{E_r^*}$ , for  $r = 2, 3, 4$ . On the other hand, we have proved in Section 4 that

$$\|f_3\|_{L^2(\Omega)} \lesssim_M \|\nabla^2 D_t h\|_{L^2(\Omega)} + \|\nabla D_t^2 h\|_{L^2(\Omega)} + \|h\|_{2,0} + \sum_{j=1,2} \|\nabla^j v\|_{L^2(\Omega)}, \quad (5.58)$$

$$\|f_4\|_{L^2(\Omega)} \lesssim_{K,M} \|\nabla^3 D_t h\|_{L^2(\Omega)} + \|\nabla^2 D_t^2 h\|_{L^2(\Omega)} + \sum_{j=2,3} \|h\|_{j,0} + \sum_{j=1,2,3} \|\nabla^j v\|_{L^2(\Omega)}, \quad (5.59)$$

$$\begin{aligned} \|f_5\|_{L^2(\Omega)} &\lesssim_{K,M} \|\nabla^3 D_t^2 h\|_{L^2(\Omega)} + \left( \sum_{j=2,3} \|\nabla^j v\|_{L^2(\Omega)} \right) \|\nabla^2 D_t^3 h\|_{L^2(\Omega)} \\ &+ \sum_{1 \leq i \leq 4} \|v\|_{i,0} + \left( \sum_{j=2,3} \|\nabla^j v\|_{L^2(\Omega)} \right) \|\nabla D_t^3 h\|_{L^2(\Omega)} + \sum_{2 \leq i \leq 4} \|h\|_{i,0}, \end{aligned} \quad (5.60)$$

and

$$\|g_3\|_{L^2(\Omega)} \lesssim_{M,c_0} W_3^*, \quad (5.61)$$

$$\|g_4\|_{L^2(\Omega)} \lesssim_{M,c_0} W_4^*, \quad (5.62)$$

$$\|g_5\|_{L^2(\Omega)} \lesssim_{K,M,c_0,\text{vol } \Omega} W_5^* + (\|\nabla^2 D_t^3 h\|_{L^2(\Omega)} + \|\nabla D_t^3 h\|_{L^2(\Omega)}) \|\sqrt{e'(h)} D_t^3 h\|_{L^2(\Omega)}. \quad (5.63)$$

Now, since  $W_{r+1}^*$  is part of  $\sqrt{E_r^*}$ , (5.25) and (5.26)-(5.27) give <sup>7</sup>

$$\|f_3\|_{L^2(\Omega)} + \|g_3\|_{L^2(\Omega)} \lesssim_{K,M,c_0,\frac{1}{\epsilon},\text{vol } \Omega} \sqrt{E_2^*}, \quad (5.64)$$

$$\|f_4\|_{L^2(\Omega)} + \|g_4\|_{L^2(\Omega)} \lesssim_{K,M,c_0,\frac{1}{\epsilon},\text{vol } \Omega} \sqrt{E_3^*}, \quad (5.65)$$

and finally

$$\|f_5\|_{L^2(\Omega)} + \|g_5\|_{L^2(\Omega)} \lesssim_{K,M,c_0,\frac{1}{\epsilon},\text{vol } \Omega} (1 + \sqrt{E_3^*})^2 \sqrt{E_4^*} \quad \text{when } \Omega \in \mathbb{R}^3 \quad (5.66)$$

<sup>7</sup>We remark here that the estimates for  $\|f_r\|$  do not require  $\|\nabla^r h\|_{L^2(\Omega)}$  norm.

## 5.6 The energy estimates

We are now ready to prove Proposition 5.1. Since we have showed that our energies  $E_r$  control the interior and boundary Sobolev norms of  $v$  and  $h$ , the only thing left is to control the product of the projected tensors, i.e.,

$$\sum_{s+k=r, s>0} \left( \sum_{0 \leq m \leq s-1} \Pi((\nabla^{m+1}v) \cdot \nabla^{s-m} D_t^k h) \right), \quad \text{for } k > 0 \quad (5.67)$$

$$\sum_{0 \leq m \leq r-2} \Pi((\nabla^{m+1}v) \cdot \nabla^{r-m} h), \quad \text{for } k = 0 \quad (5.68)$$

$$\sum_{s+k=r, s>0} \Pi((\nabla h) \cdot (\nabla^s D_t^k v)). \quad \text{for } k > 0 \quad (5.69)$$

We cannot use interpolation (A.6) here since it only applies to tangential derivative  $\bar{\nabla}$ . Our strategy is to apply Sobolev lemma (Lemma A.4) and Gagliardo-Nirenberg inequality to control terms that involving mixed derivatives<sup>8</sup>. Further, we use Theorem A.8 to control terms with full spatial derivatives.

We bound (5.67)-(5.69) when  $r = 4$  and  $\Omega \in \mathbb{R}^3$ , since other cases follow from the same method and so we omit the details. By letting  $\alpha = \nabla^{s-1}v$  in (A.18) we get

$$\|\nabla^{s-1}v\|_{L^2(\partial\Omega)} \lesssim_K \sum_{j \leq s} \|\nabla^j v\|_{L^2(\Omega)}.$$

Therefore, as we claimed in the beginning of Section 5.3,  $\langle\langle v \rangle\rangle_{r-1} \lesssim_K \sum_{i \leq r} \|v\|_{i,0}$ .

Now, each term of (5.67) is bounded as

- When  $s = 1, k = 3$  (hence  $m = 0$ )

$$\|\Pi(\nabla v \cdot \nabla D_t^3 h)\|_{L^2(\partial\Omega)} \lesssim_M \|\nabla D_t^3 h\|_{L^2(\partial\Omega)} \lesssim_{K,M,c_0,\frac{1}{\epsilon},\text{vol}\Omega} (1 + \sqrt{E_3^*}) \sqrt{E_4^*}.$$

- When  $s = 2, k = 2$  (hence  $m = 0, 1$ )

$$\begin{aligned} & \|\Pi(\nabla v \cdot \nabla^2 D_t^2 h)\|_{L^2(\partial\Omega)} + \|\Pi(\nabla^2 v \cdot \nabla D_t^2 h)\|_{L^2(\partial\Omega)} \\ & \lesssim_{K,M} \|\nabla^2 D_t^2 h\|_{L^2(\partial\Omega)} + \|\nabla^2 v\|_{L^2(\partial\Omega)}^{1/2} \|\nabla D_t^2 h\|_{L^2(\partial\Omega)}^{1/2} \|\nabla^2 v\|_{H^1}^{1/2} \|\nabla D_t^2 h\|_{H^1}^{1/2} \\ & \lesssim_{K,M,c_0,\frac{1}{\epsilon},\text{vol}\Omega} (1 + \sqrt{E_3^*})^2 \sqrt{E_4^*}. \end{aligned}$$

- When  $s = 3, k = 1$  (hence  $m = 0, 1, 2$ )

$$\begin{aligned} & \|\Pi(\nabla v \cdot \nabla^3 D_t h)\|_{L^2(\partial\Omega)} + \|\Pi(\nabla^2 v \cdot \nabla^2 D_t h)\|_{L^2(\partial\Omega)} + \|\Pi(\nabla^3 v \cdot \nabla D_t h)\|_{L^2(\partial\Omega)} \\ & \lesssim_{K,M} \|\nabla^3 D_t h\|_{L^2(\partial\Omega)} + \|\nabla^2 v\|_{L^2(\partial\Omega)}^{1/2} \|\nabla^2 D_t h\|_{L^2(\partial\Omega)}^{1/2} \|\nabla^2 v\|_{H^1}^{1/2} \|\nabla^2 D_t h\|_{H^1}^{1/2} + \|\nabla^3 v\|_{L^2(\partial\Omega)} \\ & \lesssim_{K,M,c_0,\frac{1}{\epsilon},\text{vol}\Omega} (1 + \sqrt{E_3^*})^2 \sqrt{E_4^*} \end{aligned}$$

<sup>8</sup>We want our estimates to be linear in the highest order. One can use Sobolev lemma only to control mixed Sobolev norms as well but the highest order energy would appear quadratically that way.

**Remark.** If  $\Omega \in \mathbb{R}^2$ , the terms  $\|\Pi(\nabla^2 v \cdot \nabla D_t^2 h)\|_{L^2(\partial\Omega)}$  and  $\|\Pi(\nabla^2 v \cdot \nabla^2 D_t h)\|_{L^2(\partial\Omega)}$  are bounded via Sobolev lemma. To be more specific,

$$\begin{aligned} & \|\Pi(\nabla^2 v \cdot \nabla D_t^2 h)\|_{L^2(\partial\Omega)} + \|\Pi(\nabla^2 v \cdot \nabla^2 D_t h)\|_{L^2(\partial\Omega)} \\ & \lesssim_K \left( \sum_{j \leq 2} \|\nabla^j D_t^2 h\|_{L^2(\partial\Omega)} \right) \|\nabla^2 v\|_{L^2(\partial\Omega)} + \left( \sum_{j \leq 3} \|\nabla^j D_t h\|_{L^2(\partial\Omega)} \right) \|\nabla^2 v\|_{L^2(\partial\Omega)} \lesssim (1 + \sqrt{E_3^*})^2 \sqrt{E_4^*}. \end{aligned}$$

The  $L^2$  norm of (5.68) is bounded by the same arguments used in [1]. In addition, we need the following "product rule" of the projection, in order to control intermediate terms linearly in the highest order.

**Theorem 5.4.** Let  $S, R$  be two tensors, we have

$$\Pi(S \cdot R) = \Pi(S) \cdot \Pi(R) + \Pi(S \cdot N) \tilde{\otimes} \Pi(N \cdot R), \quad (5.70)$$

where  $\tilde{\otimes}$  is the symmetric tensor product which is defined similar to the symmetric dot product.

*Proof.* (5.70) is a direct consequence of the fact  $g^{ab} = \gamma^{ab} + N^a N^b$ .  $\square$

Now if we apply (5.70) to (5.68), we get for  $0 \leq m \leq r - 2$

$$\begin{aligned} & \|\Pi(\nabla^{1+m} v \cdot \nabla^{r-m} h)\|_{L^2(\partial\Omega)} \leq \\ & \quad \|\Pi(\nabla^{1+m} v)(\Pi \nabla^{r-m} h)\|_{L^2(\partial\Omega)} + \|(\Pi N^j \nabla^{1+m} v_j)(\Pi N^j \nabla^{r-1-m} \nabla_j h)\|_{L^2(\partial\Omega)} \\ & \quad \leq \|\Pi(\nabla^{1+m} v)\|_{L^{2(r-2)/m}(\partial\Omega)} \|\Pi \nabla^{r-m} h\|_{L^{2(r-2)/(r-2-m)}(\partial\Omega)} \\ & \quad + \|\Pi(N^j \nabla^{1+m} v_j)\|_{L^{2(r-2)/m}(\partial\Omega)} \|\Pi N^j \nabla^{r-m-1} \nabla_j h\|_{L^{2(r-2)/(r-2-m)}(\partial\Omega)}. \quad (5.71) \end{aligned}$$

When  $r = 4$ , if  $m = 0$  and  $m = 2$ , this can be bounded via a priori assumptions time energies of the form  $\sqrt{E_r^*}$ , and the intermediate term  $\|\Pi(\nabla^2 v \cdot \nabla^3 h)\|_{L^2(\partial\Omega)}$  can be bounded linear in the highest orders by Theorem A.8. The first term in the second line of (5.71) is bounded by letting  $\alpha = \nabla v$  and  $\beta = \nabla^2 h$  in (A.17). To be specific,

$$\begin{aligned} & \|(\Pi \nabla^2 v)(\Pi \nabla^3 h)\|_{L^2(\partial\Omega)} \lesssim_{K,M} (|\nabla v|_{L^\infty} + \sum_{j \leq 2} \|\nabla^j v\|_{L^2(\partial\Omega)}) \|\nabla^4 h\|_{L^2(\partial\Omega)} \\ & \quad + (|\nabla^2 h|_{L^\infty} + \sum_{j \leq 3} \|\nabla^j h\|_{L^2(\partial\Omega)}) \|\nabla^3 v\|_{L^2(\partial\Omega)} \\ & \quad + (|\theta|_{L^\infty} + \|\bar{\nabla}^2 \theta\|_{L^2(\partial\Omega)}) (|\nabla v|_{L^\infty} + \sum_{j \leq 2} \|\nabla^j v\|_{L^2(\partial\Omega)}) (|\nabla^2 h|_{L^\infty} + \sum_{j \leq 3} \|\nabla^j h\|_{L^2(\partial\Omega)}). \quad (5.72) \end{aligned}$$

The term  $\|(\Pi N^j \nabla^2 v_j)(\Pi N^j \nabla^2 \nabla_j h)\|_{L^2(\partial\Omega)}$  can be bounded similarly.

As for (5.69), since  $r \leq 4$ ,  $|\nabla^s D_t^k v|$  can be bounded by

$$|\nabla^{s+1} D_t^{k-1} h| + C(K, M) \left( \sum_{j \leq r-1} |\nabla^j v| + \sum_{k+s \leq r-1, s > 0} |\partial^s D_t^k h| \right). \quad (5.73)$$

and so (5.69) can be controlled via a priori assumptions times energies.

**Proposition 5.5.** Let  $r \geq r_0 > \frac{n}{2} + \frac{3}{2}$ , there is a continuous function  $\mathcal{T}_r > 0$  such that if

$$0 < T \leq \mathcal{T}_r(c_0, K, \mathcal{E}(0), E_r^*(0), \text{vol } \Omega),$$

where

$$\mathcal{E}(t) = |(\nabla_N h(t, \cdot))^{-1}|_{L^\infty(\partial\Omega)}. \quad (5.74)$$

Then any smooth solution of (1.6) for  $0 \leq t \leq T$  satisfies

$$E_r^*(t) \leq 2E_r^*(0), \quad (5.75)$$

$$\mathcal{E}(t) \leq 2\mathcal{E}(0), \quad (5.76)$$

$$g_{ab}(0, y)Z^a Z^b \lesssim g_{ab}(t, y)Z^a Z^b \lesssim g_{ab}(0, y)Z^a Z^b, \quad (5.77)$$

and there is some fixed  $\eta > 0$  such that

$$|N(x(t, \bar{y})) - N(x(0, \bar{y}))| \lesssim \eta, \quad \bar{y} \in \partial\Omega, \quad (5.78)$$

$$|x(t, y) - x(0, y)| \lesssim \eta, \quad y \in \Omega, \quad (5.79)$$

$$\left| \frac{\partial x(t, \bar{y})}{\partial y} - \frac{\partial x(0, \bar{y})}{\partial y} \right| \lesssim \eta, \quad \bar{y} \in \partial\Omega \quad (5.80)$$

hold.

To prove Proposition 5.5, we will be using Sobolev lemmas. But then we must make sure that we can control the Sobolev constants. By Lemma A.3 and A.4, the Sobolev constants depend on  $K = \frac{1}{l_0}$ , in fact we are allowed to pick a  $K$  depending only on initial conditions, which is proved in [1]. On the other hand, the change of the Sobolev constants in time are controlled by a bound for the time derivative of the metric in Lagrangian coordinate. We also need to control the constant  $\frac{1}{\epsilon}$  appears to be in the Taylor sign condition (1.30).

**Lemma 5.6.** Assume the conditions in Proposition 5.5 hold. Then there are continuous functions  $G_{r_0}, H_{r_0}, I_{r_0}, J_{r_0}$  and  $C_{r_0}$  such that

$$\|\nabla v\|_{L^\infty(\Omega)} \leq G_{r_0}(c_0, K, E_0, \dots, E_{r_0}), \quad (5.81)$$

$$\|\nabla h\|_{L^\infty(\Omega)} \leq H_{r_0}(c_0, K, E_0, \dots, E_{r_0}, \text{vol } \Omega), \quad (5.82)$$

$$\|\nabla^2 h\|_{L^\infty(\Omega)} + \|\nabla D_t h\|_{L^\infty(\Omega)} \leq I_{r_0}(c_0, K, E_0, \dots, E_{r_0}, \text{vol } \Omega), \quad (5.83)$$

$$\|\theta\|_{L^\infty(\partial\Omega)} \leq J_{r_0}(c_0, K, \mathcal{E}, E_0, \dots, E_{r_0}, \text{vol } \Omega), \quad (5.84)$$

$$\left| \frac{d}{dt} \mathcal{E} \right| \leq C_{r_0}(c_0, K, \mathcal{E}, E_0, \dots, E_{r_0}, \text{vol } \Omega). \quad (5.85)$$

*Proof.* By Sobolev lemmas, we have

$$\|\nabla v\|_{L^\infty(\Omega)} \lesssim_K \sum_{1 \leq j \leq 3} \|\nabla^j v\|_{L^2(\Omega)},$$

$$\|\nabla h\|_{L^\infty(\Omega)} \lesssim_K \sum_{1 \leq j \leq 3} \|\nabla^j h\|_{L^2(\Omega)},$$

and

$$\|\nabla^2 h\|_{L^\infty(\Omega)} + \|\nabla D_t h\|_{L^\infty(\Omega)} \lesssim_K \sum_{2 \leq j \leq 4} \|\nabla^j h\|_{L^2(\Omega)} + \sum_{1 \leq j \leq 3} \|\nabla^j D_t h\|_{L^2(\Omega)}.$$

So, as a consequence of our interior and boundary estimates, (5.81)-(5.83) follows. On the other hand, since  $|\nabla^2 h| \geq |\Pi \nabla^2 h| = |\nabla_N h| |\theta| \geq \mathcal{E}^{-1} |\theta|$ , so (5.84) follows from (5.83). Lastly, (5.85) is a consequence of

$$\frac{d}{dt} \|(-\nabla_N h(t, \cdot))^{-1}\|_{L^\infty(\partial\Omega)} \lesssim \|(-\nabla_N h(t, \cdot))^{-1}\|_{L^\infty(\partial\Omega)}^2 \|\nabla_N h_t(t, \cdot)\|_{L^\infty(\partial\Omega)},$$

and (5.83).  $\square$

### 5.6.1 Proof of Proposition 5.5

Since when  $r \geq r_0 > \frac{n}{2} + \frac{3}{2}$ , we have

$$|\frac{d}{dt}E_r| \leq C_r(c_0, K, \mathcal{E}, E_0, \dots, E_{r_0}, \text{vol } \Omega)E_r^*,$$

and the RHS is in fact a polynomial of  $E_r^*$  with positive coefficients, we get (5.75) from Lemma 5.6 and Gronwall's lemma if  $\mathcal{T}_r(K, \mathcal{E}_0, E_r^*(0), \text{vol } \Omega) > 0$  is sufficiently small. (5.76) is a direct consequence of (5.85). In addition, we get from (5.75) and Lemma 5.6 that

$$\|\nabla v\|_{L^\infty(\Omega)} + \|\nabla h\|_{L^\infty(\Omega)} \leq C(c_0, K, \mathcal{E}(0), E_0(0), \dots, E_{r_0}(0), \text{vol } \Omega), \quad (5.86)$$

$$\|\nabla^2 h\|_{L^\infty(\Omega)} + \|\nabla D_t h\|_{L^\infty(\Omega)} + \|\theta\|_{L^\infty(\partial\Omega)} \leq C(c_0, K, \mathcal{E}(0), E_0(0), \dots, E_{r_0}(0), \text{vol } \Omega). \quad (5.87)$$

It follows from these that, when  $0 < T \leq \mathcal{T}_r(c_0, K, \mathcal{E}(0), E_r^*(0), \text{vol } \Omega)$  with  $\mathcal{T}_r$  chosen to be sufficiently small,

$$\|\nabla v(t, \cdot)\|_{L^\infty(\Omega)} + \|\nabla h(t, \cdot)\|_{L^\infty(\Omega)} \lesssim \|\nabla v(0, \cdot)\|_{L^\infty(\Omega)} + \|\nabla h(0, \cdot)\|_{L^\infty(\Omega)}, \quad (5.88)$$

and

$$\begin{aligned} \|\nabla^2 h(t, \cdot)\|_{L^\infty(\Omega)} + \|\nabla D_t h(t, \cdot)\|_{L^\infty(\Omega)} + \|\theta(t, \cdot)\|_{L^\infty(\partial\Omega)} \\ \lesssim \|\nabla^2 h(0, \cdot)\|_{L^\infty(\Omega)} + \|\nabla D_t h(0, \cdot)\|_{L^\infty(\Omega)} + \|\theta(0, \cdot)\|_{L^\infty(\partial\Omega)}, \end{aligned} \quad (5.89)$$

where  $0 < t \leq T$ .

On the other hand, we have

$$\|v(t, \cdot)\|_{L^\infty(\Omega)} \lesssim \|v(0, \cdot)\|_{L^\infty(\Omega)}, \quad (5.90)$$

$$\|\rho(t, \cdot)\|_{L^\infty(\Omega)} \lesssim \|\rho(0, \cdot)\|_{L^\infty(\Omega)}. \quad (5.91)$$

In fact, (5.90) follows since  $D_t v = -\partial h$  and (5.86), whereas (5.91) follows since  $|D_t \rho| \leq |\rho \text{div } v|$ . Now, (5.77) follows because  $D_t g$  behaves like  $\nabla v$ . Furthermore, (5.78) follows from

$$D_t N_a = -\frac{1}{2}N_a(D_t g^{cd})N_c N_d,$$

and (5.77). On the other hand, since by the definition of the Lagrangian coordinate, we have

$$D_t x(t, y) = v(t, x(t, y)),$$

and so (5.79) follows since (5.90). Lastly, because

$$D_t \frac{\partial x}{\partial y} = \frac{\partial v(t, x)}{\partial x} \frac{\partial x}{\partial y},$$

(5.80) follows since (5.81).

We close this section by briefly going over the idea which shows that one can choose  $K$  depends only on the initial conditions.

**Lemma 5.7.** Let  $0 \leq \eta \leq 2$  be a fixed number, define  $l_1 = l_1(\eta)$  to be the largest number such that

$$|N(\bar{x}_1) - N(\bar{x}_2)| \leq \eta, \quad \text{whenever } |\bar{x}_1 - \bar{x}_2| \leq l_1, \bar{x}_1, \bar{x}_2 \in \partial\mathcal{D}_t.$$

Suppose  $|\theta| \leq K$ , we recall that  $l_0$  is the injective radius defined in Section 1.3, then

$$\begin{aligned} l_0 &\geq \min(l_1/2, 1/K), \\ l_1 &\geq \min(2l_0, \eta/K). \end{aligned}$$



*Proof.* See Lemma 3.6 of [1] □

In fact, Lemma 5.7 shows that  $l_0$  and  $l_1$  are comparable to each other as long as the free surface is regular.

**Lemma 5.8.** Fix  $\eta > 0$  sufficiently small, let  $\mathcal{T}$  be in Proposition 5.5. Pick  $l_1 > 0$  such that, whenever  $|x(0, y_1) - x(0, y_2)| \leq 2l_1$ ,

$$|N(x(0, y_1)) - N(x(0, y_2))| \leq \frac{\eta}{2}. \quad (5.92)$$

Then if  $t \leq \mathcal{T}$  we have

$$|N(x(t, y_1)) - N(x(t, y_2))| \leq \eta, \quad (5.93)$$

whenever  $|x(t, y_1) - x(t, y_2)| \leq l_1$ .

*Proof.* We have

$$\begin{aligned} & |N(x(t, y_1)) - N(x(t, y_2))| \\ & \leq |N(x(t, y_1)) - N(x(0, y_1))| + |N(x(0, y_1)) - N(x(0, y_2))| + |N(x(0, y_2)) - N(x(t, y_2))|, \end{aligned} \quad (5.94)$$

and so (5.93) follows from (5.78) and (5.79). □

Lemma 5.8 allows us to pick

$$l_1(t) \leq \frac{l_1(0)}{2},$$

in other words, we have if  $\frac{1}{l_1(0)} \leq \frac{K}{2}$ , then

$$\frac{1}{l_1(t)} \leq K.$$

Therefore, Lemma 5.7 yields

$$\frac{1}{l_0(t)} \leq K.$$

## 6 Incompressible limit

In this section we prove that the energy estimates for compressible Euler equations are in fact uniform in the sound speed. For physical reasons, the sound speed is defined by

$$c(t, x)^2 := p'(\rho)$$

Let  $\{p_\kappa(\rho)\}$  be a family parametrized by  $\kappa \in \mathbb{R}^+$ , such that for each  $\kappa$  we have

$$p'_\kappa(\rho)|_{\rho=1} = \kappa.$$

We shall call  $\kappa$  as the sound speed by a slight abuse of terminology. We are concerning with fluid motion when  $\kappa$  is large and in its limit as  $\kappa \rightarrow \infty$ . We recall that the the enthalpy  $h$  has derivative

$$h'(\rho) = \frac{p'_\kappa(\rho)}{\rho},$$

and since  $p_\kappa(\rho)$  is strictly increasing for every  $\kappa$  and  $h'(\rho) > 0$ , we can write  $\rho$  as a function of  $h$  depends on  $\kappa$ . We want to impose the following conditions on  $\rho_\kappa(h)$ :

1.  $\rho_\kappa(h) \rightarrow 1$  as  $\kappa \rightarrow \infty$ .
2. Let  $e_\kappa(h) := \log \rho_\kappa(h)$ . We assume  $|e_\kappa^{(k)}(h)| \leq c_0$  for  $k \leq 6$ , where  $c_0$  is a fixed constant.
3.  $|e_\kappa^{(k)}(h)| \leq c_0 \sqrt{e'_\kappa(h)}$ , for  $k \leq 6$ .

The purpose of this section is to prove:

**Proposition 6.1.** Let  $E_{r,\kappa}$  be defined as (1.16) and for  $r \geq r_0 > \frac{n}{2} + \frac{3}{2}$ , there is a continuous function  $\mathcal{T}_r > 0$  such that if

$$0 < T \leq \mathcal{T}_r(c_0, K, \frac{1}{\epsilon}, E_{r,\kappa}^*(0), \text{vol } \Omega),$$

then any smooth solution of (1.6) for  $0 \leq t \leq T$  satisfies

$$E_{r,\kappa}^*(t) \leq 2E_{r,\kappa}^*(0). \quad (6.1)$$

**Remark.**  $\mathcal{T}_r = \mathcal{T}_r(c_0, K, \frac{1}{\epsilon}, E_{r,\kappa}^*(0), \text{vol } \Omega)$  is in fact independent of  $\kappa$ . This is because that there exists initial data so that  $E_{r,\kappa}^*(0)$  has an uniform bound (see Section 7).

Based on the analysis we have in Section 5.5, Proposition 6.1 is a direct consequence of the next theorem.

**Theorem 6.2.** Let  $E_{r,\kappa}$  be defined as (1.16), then there are continuous functions  $C_r$  such that, for  $t \in [0, T]$  and  $r \leq 4$  that

$$\left| \frac{dE_{r,\kappa}(t)}{dt} \right| \leq C_r(K, \frac{1}{\epsilon}, M, c_0, \text{vol } \mathcal{D}_t, E_{r-1,\kappa}^*) E_{r,\kappa}^*(t), \quad (6.2)$$

for all  $\kappa$ , provided that the assumptions on  $e_\kappa(h)$  hold and

$$|\theta_\kappa| + \frac{1}{l_0} \leq K, \quad \text{on } \partial\Omega, \quad (6.3)$$

$$-\nabla_N h_\kappa \geq \epsilon > 0, \quad \text{on } \partial\Omega, \quad (6.4)$$

$$1 \leq \rho_\kappa \leq M, \quad \text{in } \Omega, \quad (6.5)$$

$$|D_t^2 h_\kappa| \leq M, \quad \text{in } \Omega, \quad (6.6)$$

$$|\nabla v_\kappa| + |\nabla h_\kappa| + |\nabla^2 h_\kappa| + |\nabla D_t h_\kappa| \leq M, \quad \text{in } \Omega. \quad (6.7)$$

**Remark.** We actually do not need to assume the bound for  $|D_t h_\kappa|$ . Since  $\Omega$  is bounded and  $D_t h_\kappa = 0$  on  $\partial\Omega$ , so

$$|D_t h_\kappa|_{L^\infty} \lesssim \int_\Omega |\nabla D_t h_\kappa| \lesssim_{\text{vol } \Omega} M.$$

Together with (6.6), we have

$$|D_t h_\kappa| + |D_t^2 h_\kappa| \leq M, \quad \text{in } \Omega, \quad (6.8)$$

independent of  $\kappa$ . This is compatible with the case with fixed sound speed.

To prove Proposition 6.1, the analysis we had in the Section 5 implies that it suffices to prove that the estimates for  $\|v_\kappa\|_{r,0}$ ,  $\|h_\kappa\|_r$  and  $\langle\langle h_\kappa \rangle\rangle_r$  are bounded uniformly in  $\kappa$ . It is easy to see that under a priori assumptions (6.3)-(6.7), the estimates for  $\|f_r\|_{L^2(\Omega)}$  and  $\|g_r\|_{L^2(\Omega)}$  (Section 4) stay unchanged.

The interior estimates in Section 5.2 are uniform in the sound speed since  $\sum_{i \leq r} \|h_\kappa\|_i$  involves terms of the form  $\sum_{i \leq r} \|\sqrt{e'_\kappa(h)} D_t^i h_\kappa\|_{L^2(\Omega)}$  for each  $r$ , which means that we do not need the lower bound of  $|e'_\kappa(h)|$  in our estimates. Further, the boundary estimates for  $\sum_{i \leq r} \langle h_\kappa \rangle_i$  follows as well, which are uniform in  $\kappa$  since the interior estimates are. But we need (4.20) to estimate  $\langle h_\kappa \rangle_4$ . Finally, as for the extra a priori assumption (6.6), we can get it back by the interior estimates as in Lemma 5.6 since

$$\|D_t^2 h_\kappa\|_{L^\infty(\Omega)} \lesssim_K \sum_{j=1,2} \|\nabla^j D_t^2 h_\kappa\|_{L^2(\Omega)} + \|D_t^2 h_\kappa\|_{L^2(\Omega)}, \quad (6.9)$$

where  $\|D_t^2 h_\kappa\|_{L^2(\Omega)} \lesssim_{\text{vol } \Omega} \|\nabla D_t^2 h_\kappa\|_{L^2(\Omega)}$  via (3.3).

The above analysis shows that

$$E_{r,\kappa}^*(t) \leq 2E_{r,\kappa}^*(0),$$

regardless of the sound speed  $\kappa$ . Furthermore, since we are able to show that  $E_{r,\kappa}^*(0)$  are uniformly bounded in Section 7. A direct consequence of this is that  $v_\kappa$  and  $h_\kappa$  converge in  $C^2([0, T], \Omega)$  as  $\kappa \rightarrow \infty$ . To be more precise, we define

**Definition 6.1.** The space

$$C^l([0, T], \Omega)$$

consists all functions  $u(t, x)$  with

$$\nabla^s D_t^k u(t, \cdot), \quad s + k \leq l$$

continuous in  $\Omega$ .

**Theorem 6.3.**  $v_\kappa, h_\kappa$  converge in  $C^2([0, T] \times \Omega)$ .

*Proof.* We first show that the  $C^2$  norms of  $v_\kappa$  and  $h_\kappa$  are bounded by  $E_{r,\kappa}^*(t)$ . By the definition of  $C^2([0, T], \Omega)$ , we have

$$\|v_\kappa\|_{C^2([0, T], \Omega)}^2 + \|h_\kappa\|_{C^2([0, T], \Omega)}^2 \lesssim_K \sum_{j \leq 4} (\|v_\kappa\|_{j,0}^2 + \|h_\kappa\|_{j,0}^2) \lesssim E_{4,\kappa}^*(t) \leq 2E_{4,\kappa}^*(0),$$

by Sobolev lemma. Hence, the energy estimates (6.2) as well as the arguments in Section 7.3 yield that the quantities  $\|v_\kappa\|_{C^2([0, T], \Omega)}$  and  $\|h_\kappa\|_{C^2([0, T], \Omega)}$  are uniformly bounded. Furthermore, since the uniform bound for  $\sum_{j \leq 4} (\|v_\kappa\|_{j,0} + \|h_\kappa\|_{j,0})$  also implies that for  $s + k \leq 2$

$$\nabla^s D_t^k v_\kappa, \quad \nabla^s D_t^k h_\kappa \in C^{0, \frac{1}{2}}(\Omega),$$

in the sense of Hölder continuous functions (see [7], Chapter 5). This shows that the families  $v_\kappa$  and  $h_\kappa$  are in fact equi-continuous in  $C^2([0, T] \times \Omega)$ . Therefore,  $(v_\kappa, h_\kappa)$  convergent (after possibly passing to subsequence) by Arzela-Ascoli theorem.  $\square$

## 7 Existence of initial data satisfying the compatibility condition

In this section we show that there are initial data satisfying the compatibility conditions. We do not aim to find the largest possible set. Our purpose is to show that the analysis we done in the previous section is not about the empty set.

Let us therefore start by assuming that  $e(h) = \mathfrak{r}h$ , where  $\mathfrak{r} = \kappa^{-1}$ . We consider the wave equation

$$\mathfrak{r}D_t^2h = \Delta h + f_1, \quad h|_{t=0} = h_0, \quad D_th|_{t=0} = h_1, \quad (7.1)$$

where  $f_1 = (\partial v)(\partial v) := (\partial v)^2$ . Here we have adopted the notations that we used in Section 4. Then we must have

$$h_0|_{\partial\Omega} = h_1|_{\partial\Omega} = 0 \quad (7.2)$$

and in order for data to be compatible with  $h|_{\partial\Omega} = 0$  we must in particular have  $D_t^2h|_{\partial\Omega} = 0$  when  $t = 0$ , i.e.

$$\Delta h_0 + F_0 = 0, \quad \text{on } \partial\Omega \quad (7.3)$$

where  $F_0 = f_1|_{t=0}$ . This could be achieved by solving for (7.3) equal to 0 everywhere. Similarly, by differentiating the equation we get

$$\mathfrak{r}D_t^3h = \Delta D_th + f_2. \quad (7.4)$$

We also need

$$\Delta h_1 + F_1 = 0, \quad \text{on } \partial\Omega \quad (7.5)$$

where  $F_1 = f_2|_{t=0}$ , and this could also be achieved by solving for this equal to 0 everywhere. However, the above procedure does not lead to that the higher order compatibility conditions are satisfied. To get more compatibility conditions satisfied one needs to solve for

$$\Delta h_2 + F_2 = 0, \quad \text{in } \Omega \quad (7.6)$$

where  $h_2 = D_t^2h|_{t=0}$  and  $F_2 = f_3|_{t=0}$ , with boundary condition

$$h_2|_{\partial\Omega} = 0, \quad (7.7)$$

in which case we then can solve for

$$\Delta h_0 + F_0 = \mathfrak{r}h_2, \quad h_0|_{\partial\Omega} = 0. \quad (7.8)$$

This procedure works to find data that satisfy finitely many compatibility conditions. Moreover, if  $\mathfrak{r} = 0$  it clearly reduces to the incompressible data.

In general we choose data that solves

$$\Delta h_k + F_k = 0, \quad h_k|_{\partial\Omega} = 0, \quad k = N, N-1, \quad (7.9)$$

and then inductively

$$\Delta h_k + F_k = \mathfrak{r}h_{k+2}, \quad h_k|_{\partial\Omega} = 0, \quad k = N-2, \dots, 0, \quad (7.10)$$

where  $h_k = D_t^k h|_{t=0}$ ,  $F_k = f_{k+1}|_{t=0}$ .

### 7.1 The a priori energy bounds for the full system

Our energy estimate requires that the compatibility conditions to be satisfied up to 5th order, i.e., we need to find  $D_t^k h|_{t=0} = h_k$  such that

$$h_k|_{\partial\Omega} = 0, \quad 0 \leq k \leq 5. \quad (7.11)$$

This can be achieved by solving

$$\begin{cases} \Delta h_k = F_k + \mathfrak{r}h_{k+2}, & \text{in } \Omega \\ h_4 = h_5 = 0, & \text{in } \Omega \\ h_k|_{\partial\Omega} = 0, & \text{on } \partial\Omega \end{cases} \quad (7.12)$$

for  $0 \leq k \leq 3$ . Here,

$$F_3 = c_{\alpha_1 \dots \alpha_m \beta_1 \dots \beta_n}^{\gamma_1 \dots \gamma_n} (\partial^{\alpha_1} v_0) \dots (\partial^{\alpha_m} v_0) (\partial^{\beta_1} h_{\gamma_1}) \dots (\partial^{\beta_n} h_{\gamma_n}) + d(\partial v_0)^5, \quad (7.13)$$

where

$$\begin{aligned} \alpha_1 + \dots + \alpha_m + (\beta_1 + \gamma_1) + \dots + (\beta_n + \gamma_n) &= 5, \\ 1 \leq \alpha_i \leq 3, \quad 1 \leq \beta_j + \gamma_j \leq 4, \quad 1 \leq i \leq m, \quad 1 \leq j \leq n, \\ 1 \leq m + n \leq 5. \end{aligned}$$

Similarly,

$$F_2 = c_{\alpha_1 \dots \alpha_m \beta_1 \dots \beta_n}^{\gamma_1 \dots \gamma_n} (\partial^{\alpha_1} v_0) \dots (\partial^{\alpha_m} v_0) (\partial^{\beta_1} h_{\gamma_1}) \dots (\partial^{\beta_n} h_{\gamma_n}) + d(\partial v_0)^4, \quad (7.14)$$

where

$$\begin{aligned} \alpha_1 + \dots + \alpha_m + (\beta_1 + \gamma_1) + \dots + (\beta_n + \gamma_n) &= 4, \\ 1 \leq \alpha_i \leq 2, \quad 1 \leq \beta_j + \gamma_j \leq 3, \quad 1 \leq i \leq m, \quad 1 \leq j \leq n, \\ 1 \leq m + n \leq 4. \end{aligned}$$

and

$$F_1 = c_{\alpha, \beta} (\partial^\alpha v_0) (\partial^\beta h_0), \quad 1 \leq \alpha, \beta \leq 2, \quad \alpha + \beta = 3, \quad (7.15)$$

$$F_0 = (\partial v_0)^2. \quad (7.16)$$

We show the existence of solution for (7.12) by successive approximation starting from the solution  $(h_0^0, h_1^0, h_2^0, h_3^0)$  that solves

$$\Delta h_k^0 = F_k(\partial^\alpha v_0, \partial^{\beta_0} h_0^0, \dots, \partial^{\beta_{k-1}} h_{k-1}^0), \quad k = 0, 1, 2, 3 \quad (7.17)$$

and we define  $(h_0^\nu, \dots, h_3^\nu)$  inductively by solving

$$\begin{cases} \Delta h_3^\nu = F_3(\partial^\alpha v_0, \partial^{\beta_0} h_0^\nu, \partial^{\beta_1} h_1^\nu, \partial^{\beta_2} h_2^\nu), \\ \Delta h_2^\nu = F_2(\partial^\alpha v_0, \partial^{\beta_0} h_0^\nu, \partial^{\beta_1} h_1^\nu), \\ \Delta h_1^\nu = \mathfrak{r}h_3^{\nu-1} + F_1(\partial^\alpha v_0, \partial^{\beta_0} h_0^\nu), \\ \Delta h_0^\nu = \mathfrak{r}h_2^{\nu-1} + (\partial v_0)(\partial v_0). \end{cases} \quad (7.18)$$

We define that for  $0 \leq k \leq 3$ ,

$$m_k^\nu := \|h_k^\nu\|_{H^{s-k}(\Omega)}, \quad s \geq 5$$

$$m_*^\nu := \sum_k m_k^\nu.$$

We shall apply the standard elliptic estimate as well as Sobolev lemmas to get bounds for  $m_k^\nu$ . It is worth mentioning that since we are working in dimensions 2 and 3,  $H^s$  is an algebra for  $s \geq 2$ , i.e.,

$$\|fg\|_{H^s} \leq C\|f\|_{H^s}\|g\|_{H^s}, \quad s \geq 2, \quad (7.19)$$

which is a direct consequence of Sobolev lemma (A.6). On the other hand, Sobolev lemma (A.5) implies

$$\|u_1 \cdots u_n\|_{H^s} \leq C\|u_1\|_{H^{s+1}} \cdots \|u_n\|_{H^{s+1}}, \quad n = 2, 3. \quad (7.20)$$

Now, since  $s \geq 4$  implies that  $H^{s-2}$  is an algebra, we have

$$m_0^\nu \leq C\mathfrak{r}\|h_2^{\nu-1}\|_{H^{s-2}} + C\|(\partial v_0)^2\|_{H^{s-2}} \leq C\mathfrak{r}m_2^{\nu-1} + C\|v_0\|_{H^s}^2, \quad (7.21)$$

and

$$\begin{aligned} m_1^\nu &\leq C\mathfrak{r}\|h_3^{\nu-1}\|_{H^{s-3}} + C\|F_1\|_{H^{s-2}} \\ &\leq C\mathfrak{r}m_3^{\nu-1} + C\|v_0\|_{H^s}\|h_0\|_{H^s} \\ &= C\mathfrak{r}m_3^{\nu-1} + C\|v_0\|_{H^s}m_0^\nu. \end{aligned} \quad (7.22)$$

However, the estimates for  $m_2^\nu$  and  $m_3^\nu$  are a bit more complicated. The standard elliptic estimate yields that

$$\begin{aligned} m_2^\nu &\leq C\|F_2\|_{H^{s-4}}, \\ m_3^\nu &\leq C\|F_3\|_{H^{s-5}}, \end{aligned}$$

and the following analysis is devoted to bound  $\|F_2\|_{H^{s-4}}$  and  $\|F_3\|_{H^{s-5}}$ .

### 7.1.1 Bounds for $\|F_2\|_{H^{s-4}}$

Since  $F_2$  is a sum of products of the form (7.14), and each product involves at least 2 but no more than 4 terms, we have

- If the product involves less than 4 terms, i.e., it is of the form

$$(\partial^{\alpha_1} v_0) \cdots (\partial^{\alpha_m} v_0) (\partial^{\beta_1} h_{\gamma_1}) \cdots (\partial^{\beta_n} h_{\gamma_n}), \quad m+n \leq 3,$$

and for each  $i, j$ , we have  $1 \leq \alpha_i \leq 2$ ,  $1 \leq (\beta_j + \gamma_j) \leq 3$  as well as

$$\alpha_1 + \cdots + \alpha_n + (\beta_1 + \gamma_1) + \cdots + (\beta_n + \gamma_n) = 4.$$

This guarantees that only

$$\begin{aligned} \partial^\alpha v_0, \quad 1 \leq \alpha \leq 2, \\ \partial^{\beta_k} h_k, \quad 1 \leq \beta_k \leq 3-k, \quad k=0,1 \end{aligned}$$

are allowed in the product. Therefore, by (7.20), we get

$$\begin{aligned}
 & \|(\partial^{\alpha_1} v_0) \cdots (\partial^{\alpha_m} v_0)(\partial^{\beta_1} h_{\gamma_1}) \cdots (\partial^{\beta_n} h_{\gamma_n})\|_{H^{s-4}} \\
 & \leq C \| \partial^{\alpha_1} v_0 \|_{H^{s-3}} \cdots \| \partial^{\alpha_m} v_0 \|_{H^{s-3}} \| \partial^{\beta_1} h_{\gamma_1} \|_{H^{s-3}} \cdots \| \partial^{\beta_n} h_{\gamma_n} \|_{H^{s-3}} \\
 & \leq Cp_2(\|v_0\|_{H^s}, \|h_0\|_{H^s}, \|h_1\|_{H^{s-1}}),
 \end{aligned} \tag{7.23}$$

for some polynomial  $p_2$ .

- If the product involves exactly 4 terms, then we must have

$$\alpha_i \leq 1, \quad \beta_j \leq 1, \quad \gamma_j = 0.$$

Hence, by the Sobolev estimate (7.19),

$$\begin{aligned}
 & \|(\partial^{\alpha_1} v_0) \cdots (\partial^{\alpha_m} v_0)(\partial^{\beta_1} h_0) \cdots (\partial^{\beta_n} h_0)\|_{H^{s-4}} \\
 & \leq C \|(\partial^{\alpha_1} v_0) \cdots (\partial^{\alpha_m} v_0)(\partial^{\beta_1} h_0) \cdots (\partial^{\beta_n} h_0)\|_{H^{s-2}} \\
 & \leq C \|v_0\|_{H^s}^m \|h_0\|_{H^s}^n.
 \end{aligned} \tag{7.24}$$

Therefore, we conclude that

$$m_2^\nu \leq CP_2(\|v_0\|_{H^s}, m_0^\nu, m_1^\nu), \tag{7.25}$$

for some polynomial  $P_2$ .

### 7.1.2 Bounds for $\|F_3\|_{H^{s-5}}$

We shall proceed as in the previous case. Since  $F_5$  is a sum of products of the form (7.13), and each product involves at least 2 but no more than 5 terms, we have

- If the product involves no more than 3 terms, i.e., it is of the form

$$(\partial^{\alpha_1} v_0) \cdots (\partial^{\alpha_m} v_0)(\partial^{\beta_1} h_{\gamma_1}) \cdots (\partial^{\beta_n} h_{\gamma_n}), \quad 1 \leq m+n \leq 3,$$

and for each  $i, j$ , we have  $1 \leq \alpha_i \leq 3$ ,  $1 \leq (\beta_j + \gamma_j) \leq 4$  as well as

$$\alpha_1 + \cdots + \alpha_n + (\beta_1 + \gamma_1) + \cdots + (\beta_n + \gamma_n) = 5.$$

This implies that only

$$\begin{aligned}
 & \partial^\alpha v_0, \quad 1 \leq \alpha \leq 3, \\
 & \partial^{\beta_k} h_k, \quad 1 \leq \beta_k \leq 4 - k, \quad k = 0, 1, 2
 \end{aligned}$$

are allowed to be included in the product. Therefore, by (7.20), we have

$$\begin{aligned}
 & \|(\partial^{\alpha_1} v_0) \cdots (\partial^{\alpha_m} v_0)(\partial^{\beta_1} h_{\gamma_1}) \cdots (\partial^{\beta_n} h_{\gamma_n})\|_{H^{s-5}} \\
 & \leq C \| \partial^{\alpha_1} v_0 \|_{H^{s-4}} \cdots \| \partial^{\alpha_m} v_0 \|_{H^{s-4}} \| \partial^{\beta_1} h_{\gamma_1} \|_{H^{s-4}} \cdots \| \partial^{\beta_n} h_{\gamma_n} \|_{H^{s-4}} \\
 & \leq Cp_3(\|v_0\|_{H^s}, \|h_0\|_{H^s}, \|h_1\|_{H^{s-1}}, \|h_2\|_{H^{s-2}}),
 \end{aligned} \tag{7.26}$$

for some polynomial  $p_3$ .

- If the product involves exactly 4 terms, then for each  $i, j$ ,

$$\begin{aligned} 1 &\leq \alpha_i \leq 2, \\ 1 &\leq \beta_j + \gamma_j \leq 2, \end{aligned}$$

and thus only

$$\begin{aligned} &\partial^\alpha v_0, \quad 1 \leq \alpha \leq 2, \\ &\partial^{\beta_k} h_k, \quad 1 \leq \beta_k \leq 2 - k, \quad k = 0, 1 \end{aligned}$$

are allowed in each product. Hence one can then use Sobolev estimate (7.19) to bound the product in  $H^{s-2}$  norm, i.e.,

$$\|(\partial^{\alpha_1} v_0) \cdots (\partial^{\alpha_m} v_0)(\partial^{\beta_1} h_{\gamma_1}) \cdots (\partial^{\beta_n} h_{\gamma_n})\|_{H^{s-2}} \leq C p_3(\|v_0\|_{H^s}, \|h_0\|_{H^s}, \|h_1\|_{H^{s-1}}). \quad (7.27)$$

- If the product involves exactly 5 terms, then only  $\partial v_0$  and  $\partial h_0$  are allowed in each product. Hence,

$$\|(\partial^{\alpha_1} v_0) \cdots (\partial^{\alpha_m} v_0)(\partial^{\beta_1} h_{\gamma_1}) \cdots (\partial^{\beta_n} h_{\gamma_n})\|_{H^{s-2}} \leq C \|v_0\|_{H^s}^m \|h_0\|_{H^s}^n. \quad (7.28)$$

Therefore, we conclude that

$$m_3^\nu \leq C P_3(\|v_0\|_{H^s}, m_0^\nu, m_1^\nu, m_2^\nu), \quad (7.29)$$

for some polynomial  $P_3$ .

Summing up the estimates for  $m_k^\nu$  for  $0 \leq k \leq 3$ , we have

$$m_*^\nu \leq P(C, \mathfrak{r} m_*^{\nu-1}, \|v_0\|_{H^s}), \quad (7.30)$$

for some polynomial  $P$ . In particular, this implies that  $m_*^0$  is uniformly bounded by a function depends on  $\|v_0\|_{H^s}$  by taking  $\nu = 0$ . Therefore,  $m_*^\nu$  are uniformly bounded for all  $\nu$  by induction if  $\mathfrak{r}$  is chosen to be sufficiently small.

## 7.2 The iteration scheme

Let's define

$$\begin{aligned} A_k^\nu &:= h_k^\nu - h_k^{\nu-1}, \\ M_k^\nu &:= \|A_k^\nu\|_{H^s(\Omega)}, \quad s \geq 5 \\ M_*^\nu &:= \sum_k M_k^\nu. \end{aligned}$$

Now, we subtract two successive systems of (7.18) and obtain

$$\begin{cases} \Delta A_3^\nu = F_3(\partial^\alpha v_0, \partial^{\beta_0} h_0^\nu, \partial^{\beta_1} h_1^\nu, \partial^{\beta_2} h_2^\nu) - F_3(\partial^\alpha v_0, \partial^{\beta_0} h_0^{\nu-1}, \partial^{\beta_1} h_1^{\nu-1}, \partial^{\beta_2} h_2^{\nu-1}), \\ \Delta A_2^\nu = F_2(\partial^\alpha v_0, \partial^{\beta_0} h_0^\nu, \partial^{\beta_1} h_1^\nu) - F_2(\partial^\alpha v_0, \partial^{\beta_0} h_0^{\nu-1}, \partial^{\beta_1} h_1^{\nu-1}), \\ \Delta A_1^\nu = \mathfrak{r} A_3^{\nu-1} + F_1(\partial^\alpha v_0, \partial^{\beta_0} h_0^\nu) - F_1(\partial^\alpha v_0, \partial^{\beta_0} h_0^{\nu-1}), \\ \Delta A_0^\nu = \mathfrak{r} A_2^{\nu-1}. \end{cases} \quad (7.31)$$



Here,

$$\begin{aligned}
 & F_3(\partial^\alpha v_0, \partial^{\beta_0} h_0^\nu, \partial^{\beta_1} h_1^\nu, \partial^{\beta_2} h_2^\nu) - F_3(\partial^\alpha v_0, \partial^{\beta_0} h_0^{\nu-1}, \partial^{\beta_1} h_1^{\nu-1}, \partial^{\beta_2} h_2^{\nu-1}) \\
 &= C_{\alpha_1 \dots \alpha_m \beta_1 \dots \beta_n}^{\gamma_1 \dots \gamma_n} \left( (\partial^{\alpha_1} v_0) \dots (\partial^{\alpha_m} v_0) (\partial^{\beta_1} A_{\gamma_1}^\nu) \dots (\partial^{\beta_n} h_{\gamma_n}^\nu) \right) \\
 &+ \dots + \left( (\partial^{\alpha_1} v_0) \dots (\partial^{\alpha_m} v_0) (\partial^{\beta_1} h_{\gamma_1}^{\nu-1}) \dots (\partial^{\beta_n} A_{\gamma_n}^\nu) \right),
 \end{aligned} \tag{7.32}$$

where

$$\begin{aligned}
 & \alpha_1 + \dots + \alpha_m + (\beta_1 + \gamma_1) + \dots + (\beta_n + \gamma_n) = 5, \\
 & 1 \leq \alpha_i \leq 3, \quad 1 \leq \beta_j + \gamma_j \leq 4, \quad 1 \leq i \leq m, \quad 1 \leq j \leq n, \\
 & 1 \leq m + n \leq 5.
 \end{aligned}$$

Similarly,

$$\begin{aligned}
 & F_2(\partial^\alpha v_0, \partial^{\beta_0} h_0^\nu, \partial^{\beta_1} h_1^\nu) - F_2(\partial^\alpha v_0, \partial^{\beta_0} h_0^{\nu-1}, \partial^{\beta_1} h_1^{\nu-1}) \\
 &= C_{\alpha_1 \dots \alpha_m \beta_1 \dots \beta_n}^{\gamma_1 \dots \gamma_n} \left( (\partial^{\alpha_1} v_0) \dots (\partial^{\alpha_m} v_0) (\partial^{\beta_1} A_{\gamma_1}^\nu) \dots (\partial^{\beta_n} h_{\gamma_n}^\nu) \right) \\
 &+ \dots + \left( (\partial^{\alpha_1} v_0) \dots (\partial^{\alpha_m} v_0) (\partial^{\beta_1} h_{\gamma_1}^{\nu-1}) \dots (\partial^{\beta_n} A_{\gamma_n}^\nu) \right),
 \end{aligned} \tag{7.33}$$

$$\begin{aligned}
 & \alpha_1 + \dots + \alpha_m + (\beta_1 + \gamma_1) + \dots + (\beta_n + \gamma_n) = 4, \\
 & 1 \leq \alpha_i \leq 2, \quad 1 \leq \beta_j + \gamma_j \leq 3, \quad 1 \leq i \leq m, \quad 1 \leq j \leq n, \\
 & 1 \leq m + n \leq 4.
 \end{aligned}$$

and

$$F_1(\partial^\alpha v_0, \partial^\beta h_0^\nu) - F_1(\partial^\alpha v_0, \partial^\beta h_0^{\nu-1}) = c_{\alpha\beta}(\partial^\alpha v_0)(\partial^\beta A_0^\nu), \quad 1 \leq \alpha, \beta \leq 2, \quad \alpha + \beta = 3. \tag{7.34}$$

Hence, the same analysis which we applied to bound  $m_k^\nu$  yields

$$M_0^\nu \leq C \mathfrak{r} M_2^{\nu-1}, \tag{7.35}$$

$$M_1^\nu \leq C \mathfrak{r} M_3^{\nu-1} + C \|v_0\|_{H^s} M_0^\nu, \tag{7.36}$$

$$M_2^\nu \leq C q_2 (m_*^\nu + m_*^{\nu-1} + \|v_0\|_{H^s}) (M_1^\nu + M_0^\nu), \tag{7.37}$$

$$M_3^\nu \leq C q_3 (m_*^\nu + m_*^{\nu-1} + \|v_0\|_{H^s}) (M_2^\nu + M_1^\nu + M_0^\nu), \tag{7.38}$$

for some polynomials  $q_2$  and  $q_3$ .

Summing these up, we have

$$M_*^\nu \leq \mathfrak{r} q(C, m, \|v_0\|_{H^s}) M_*^{\nu-1}, \tag{7.39}$$

for some polynomial  $q$ , where

$$m_*^\nu \leq m := m(C, \|v_0\|_{H^s})$$

for each  $\nu$ . This implies inductively that

$$M_*^\nu \leq \left( \mathfrak{r} q(C, m, \|v_0\|_{H^{s-1}}) \right)^\nu M_*^0. \tag{7.40}$$

But since  $M_*^0 = m_*^0$ , and so if  $\mathfrak{r}$  is chosen such that

$$\mathfrak{r} q(C, m, \|v_0\|_{H^{s-1}}) < 1,$$

then it is easy to see that

$$M_*^\nu + \dots + M_*^{\nu+n} \rightarrow 0$$

as  $\nu, n \rightarrow \infty$ .

**Remark.** Our method is systematic and so we can solve for data that satisfies  $N$ -compatibility conditions for any finite  $N$ . This, together with the fact that one can also generalize the energy estimate (5.1) to any order, the result of [13] guarantees the existence of solution for the Euler equations within  $[0, T]$  for each  $\kappa$ , which converges to the solution for the incompressible Euler equations as  $\kappa \rightarrow \infty$ .

### 7.3 Uniform bounds for $E_{4,\kappa}^*(0)$

We are now able to show  $E_{4,\kappa}^*(0)$  in Section 6 is uniformly bounded regardless of  $\kappa$ . This is because that

$$\sum_{k+s \leq 4} \int_{\Omega} \rho_0 Q(\partial^s h_k, \partial^s h_k) dx \lesssim \sum_{0 \leq k \leq 4} \|h_k\|_{H^{4-k}(\Omega)}^2 \leq m, \quad (7.41)$$

and by the trace lemma,

$$\sum_{k+s \leq 4} \int_{\partial\Omega} \rho_0 Q(\partial^s h_k, \partial^s h_k) dS \lesssim \sum_{0 \leq k \leq 4} \|h_k\|_{H^{4-k}(\partial\Omega)}^2 \lesssim \sum_{0 \leq k \leq 4} \|h_k\|_{H^{5-k}(\Omega)}^2 \leq m. \quad (7.42)$$

In addition to these, we have

$$\sum_{0 \leq k \leq 4} \|(\partial^{4-k} D_t^k v)|_{t=0}\|_{L^2} \lesssim \|v_0\|_{H^4} + \sum_{0 \leq k \leq 3} \|h_k\|_{H^{4-k}}, \quad (7.43)$$

since  $D_t v = -\partial h$ . This shows

$$\sum_{k+s \leq 4} \int_{\Omega} \rho_0 Q(\partial^s D_t^k v|_{t=0}, \partial^s D_t^k v|_{t=0}) dx \quad (7.44)$$

is uniformly bounded as well. Finally, since  $h_4 = h_5 = 0$  in  $\Omega$ , we have

$$\sum_{k=2}^5 W_k(0) \lesssim m,$$

and hence we have  $E_{4,\kappa}^*(0)$  bounded uniformly.

### 7.4 The Taylor sign condition

Since  $\mathcal{D}_0$  is bounded, we assume the Taylor sign condition holds when  $t = 0$ , i.e.,

$$\nabla_N h_0 \leq -\epsilon < 0, \quad \text{on } \partial\mathcal{D}_0. \quad (7.45)$$

This will be true under small perturbation in  $[0, T]$  due to (5.85). Given any data for the incompressible equations  $v_0$  that satisfy the sign condition our data for the compressible equations  $h_0$  will also satisfy (7.45) if  $\mathfrak{r}$  is sufficiently small. In fact, since

$$\Delta h_0 = (\partial v_0)^2 + \mathfrak{r} h_2,$$

and so

$$\Delta(h_0 - p_0) = \mathfrak{r} h_2.$$

On the other hand, the standard elliptic estimates yield

$$\|h_0 - p_0\|_{H^s} \leq \mathfrak{r} \|h_2\|_{H^{s-2}}.$$

However, we have  $\|h_0 - p_0\|_{C^1} \leq \|h_0 - p_0\|_{H^s}$  for any  $s > n/2 + 1$  and in our case  $n \leq 4$ , but since  $s \geq 5$ , and so we have  $\|h_2\|_{H^2} \leq C$  independent of  $\mathfrak{r}$ .

## A Appendix

### List of notations:

- $D_t$ : the material derivative
- $\partial_i$ : partial derivative with respect to Eulerian coordinate  $x_i$
- $\mathcal{D}_t \in \mathbb{R}^n$ : the domain occupied by fluid particles at time  $t$  in Eulerian coordinate
- $\Omega \in \mathbb{R}^n$ : the domain occupied by fluid particles in Lagrangian coordinate
- $\partial_a = \frac{\partial}{\partial y_a}$ : partial derivative with respect to Lagrangian coordinate  $y_a$
- $\nabla_a$ : covariant derivative with respect to  $y_a$
- $\Pi S$ : projected tensor  $S$  on the boundary
- $\bar{\nabla}, \bar{\partial}$ : projected derivative on the boundary
- $N$ : the outward unit normal of the boundary
- $\theta = \bar{\nabla} N$ : the second fundamental form of the boundary
- $\sigma = tr(\theta)$ : the mean curvature

### Mixed norms

- $\langle \langle \cdot \rangle \rangle_r = \sum_{k+s=r} \|\nabla^s D_t^k \cdot\|_{L^2(\partial\Omega)}$
- $\|\cdot\|_{r,0} = \sum_{s+k=r, k < r} \|\nabla^s D_t^k \cdot\|_{L^2(\Omega)}$
- $\|\cdot\|_r = \|\cdot\|_{r,0} + \|\sqrt{e'(h)} D_t^r \cdot\|_{L^2(\Omega)}$

### A.1 Covariant differentiation in the Lagrangian coordinate

The covariant differentiation of a  $(0, r)$  tensor  $k(t, y)$  is the  $(0, r+1)$  tensor given by

$$\nabla_a k_{a_1, \dots, a_r} = \frac{\partial k_{a_1, \dots, a_r}}{\partial y^a} - \Gamma_{aa_1}^d k_{d, \dots, a_r} - \Gamma_{aa_r}^d k_{a_1, \dots, d},$$

where the Christoffel symbols  $\Gamma_{ab}^c$  is given by

$$\Gamma_{ab}^c = \frac{g^{cd}}{2} \left( \frac{\partial g_{bd}}{\partial y^a} + \frac{\partial g_{ad}}{\partial y^b} - \frac{\partial g_{ab}}{\partial y^d} \right) = \frac{\partial^2 x^i}{\partial y^a \partial y^b} \frac{\partial y^c}{\partial x^i},$$

where

$$g_{ab}(t, y) = \delta_{ij} \frac{\partial x^i}{\partial y^a} \frac{\partial x^j}{\partial y^b},$$

and  $g^{cd}$  is the inverse of  $g_{ab}$ . The second equality is deduced by letting  $v(x)$  be a tangent vector expressed in  $x$ -coordinate and  $u(y)$  be the same vector expressed in  $y$ -coordinate, then

$$\frac{\partial u_a(y)}{\partial y^b} = \frac{\partial^2 x^i}{\partial y^a \partial y^b} \frac{\partial y^c}{\partial x^i} v_c(y) + \frac{\partial x^i}{\partial y^a} \frac{\partial x^j}{\partial y^b} \frac{\partial v_i(x)}{\partial x^j}.$$

and so  $\Gamma_{ab}^c = \frac{\partial^2 x^i}{\partial y^a \partial y^b} \frac{\partial y^c}{\partial x^i}$  follows from the definition of the covariant differentiation. If  $w(t, x)$  is the  $(0, r)$  tensor expressed in the  $x$ -coordinate, then the same tensor  $k(t, y)$  expressed in  $y$ -coordinate is given by

$$k_{a_1, \dots, a_r}(t, y) = \frac{\partial x^{i_1}}{\partial y^{a_1}} \cdots \frac{\partial x^{i_r}}{\partial y^{a_r}} w_{i_1, \dots, i_r}(t, x),$$

and by the transformation property of tensors,

$$\nabla_a k_{a_1, \dots, a_r} = \frac{\partial x^i}{\partial y^a} \frac{\partial x^{i_1}}{\partial y^{a_1}} \cdots \frac{\partial x^{i_r}}{\partial y^{a_r}} \frac{\partial w_{i_1, \dots, i_r}}{\partial x^i}.$$

Covariant derivative is constructed so the norms of tensors are invariant under change of coordinates,

$$g^{a_1 b_1} \cdots g^{a_r b_r} k_{a_1, \dots, a_r} k_{b_1, \dots, b_r} = \delta^{i_1 j_1} \cdots \delta^{i_r j_r} w_{i_1, \dots, i_r} w_{j_1, \dots, j_r}.$$

## A.2 Extension of normal to the interior and the geodesic normal coordinate

The definition of our energy (1.16) relies on extending the normal to the interior, which is done by foliating the domain close to the boundary into the surface that do not self-intersect. We also want to control the time evolution of the boundary, which can be measured by the time derivative of the normal in the Lagrangian coordinate. We conclude the above statements by the following two lemmas, whose proof can be found in [1].

**Lemma A.1.** let  $l_0$  be the injective radius (1.14), and let  $d(y) = \text{dist}_g(y, \partial\Omega)$  be the geodesic distance in the metric  $g$  from  $y$  to  $\partial\Omega$ . Then the co-normal  $n = \nabla d$  to the set  $S_a = \partial\{y \in \Omega : d(y) = a\}$  satisfies, when  $d(y) \leq \frac{l_0}{2}$  that

$$|\nabla n| \lesssim |\theta|_{L^\infty(\partial\Omega)}, \quad (\text{A.1})$$

$$|D_t n| \lesssim |D_t g|_{L^\infty(\Omega)}, \quad (\text{A.2})$$

where we have used the convention that  $A \lesssim B$  means  $A \leq CB$  for universal constant  $C$ .

**Lemma A.2.** let  $l_0$  be the injective radius (1.14), and let  $d_0$  be a fixed number such that  $\frac{l_0}{16} \leq d_0 \leq \frac{l_0}{2}$ . Let  $\eta$  be a smooth cut-off function satisfying  $0 \leq \eta(d) \leq 1$ ,  $\eta(d) = 1$  when  $d \leq \frac{d_0}{4}$  and  $\eta(d) = 0$  when  $d > \frac{d_0}{2}$ . Then the pseudo-Riemannian metric  $\gamma$  given by

$$\gamma_{ab} = g_{ab} - \tilde{n}_a \tilde{n}_b,$$

where  $\tilde{n}_c = \eta(\frac{d}{d_0}) \nabla_c d$  satisfies

$$|\nabla \gamma|_{L^\infty(\Omega)} \lesssim (|\theta|_{L^\infty(\partial\Omega)} + \frac{1}{l_0}) \quad (\text{A.3})$$

$$|D_t \gamma(t, y)| \lesssim |D_t g|_{L^\infty(\Omega)}. \quad (\text{A.4})$$

**Remark.** The above two lemmas yield that the quantities  $|D_t n|$  and  $|D_t \gamma(t, y)|$  involved in the  $Q$ -inner product is controlled by the a priori assumptions (1.29)-(1.33), since  $D_t g$  behaves like  $\nabla v$  by (2.7). Hence, the time derivative on the coefficients of the  $Q$ -inner product generates only lower-order terms. In addition, by (1.29),  $|\nabla n|$  and  $|\nabla \gamma|$  are controlled by  $K$ , which is essential when proving the elliptic estimates.

### A.3 Sobolev lemmas

Let us now state some Sobolev lemmas in a domain with boundary.

**Lemma A.3.** (Interior Sobolev inequalities) Suppose  $\frac{1}{l_0} \leq K$  and  $\alpha$  is a  $(0, r)$  tensor, then

$$\|\alpha\|_{L^{\frac{2n}{n-2s}}(\Omega)} \lesssim_K \sum_{l=0}^s \|\nabla^l \alpha\|_{L^2(\Omega)}, \quad 2s < n, \quad (\text{A.5})$$

$$\|\alpha\|_{L^\infty(\Omega)} \lesssim_K \sum_{l=0}^s \|\nabla^l \alpha\|_{L^2(\Omega)}, \quad 2s > n. \quad (\text{A.6})$$

Similarly, on  $\partial\Omega$ , we have

**Lemma A.4.** (Boundary Sobolev inequalities)

$$\|\alpha\|_{L^{\frac{2(n-1)}{n-1-2s}}(\partial\Omega)} \lesssim_K \sum_{l=0}^s \|\nabla^l \alpha\|_{L^2(\partial\Omega)}, \quad 2s < n-1, \quad (\text{A.7})$$

$$\|\alpha\|_{L^\infty(\partial\Omega)} \lesssim_K \delta \|\nabla^s \alpha\|_{L^2(\partial\Omega)} + \delta^{-1} \sum_{l=0}^{s-1} \|\nabla^l \alpha\|_{L^2(\partial\Omega)}, \quad 2s > n-1, \quad (\text{A.8})$$

for any  $\delta > 0$ . In addition, for the boundary we can also interpret the norm be given by the inner product  $\langle \alpha, \alpha \rangle = \gamma^{IJ} \alpha_I \alpha_J$ , and the covariant derivative is then given by  $\bar{\nabla}$ .

### A.4 Interpolation on spatial derivatives

We shall first record spatial interpolation inequalities. Most of the results are standard in  $\mathbb{R}^n$ , but we must control how it depends on the geometry of our evolving domain. The coefficients involved in our inequalities depend on  $K$ , whose reciprocal is the lower bound for the injective radius  $l_0$ . We omit the proofs, which can be found in the appendix of [1].

**Theorem A.5.** (Interior interpolation) Let  $u$  be a  $(0, r)$  tensor, and suppose  $\frac{1}{l_0} \leq K$ , we have

$$\sum_{j=0}^l \|\nabla^j u\|_{L^{\frac{2r}{k}}(\Omega)} \lesssim \|u\|_{L^{\frac{2(r-l)}{k-l}}(\Omega)}^{1-\frac{l}{r}} \left( \sum_{i=0}^r \|\nabla^i u\|_{L^2(\Omega)} K^{r-i} \right)^{\frac{l}{r}}. \quad (\text{A.9})$$

In particular, if  $k = l$ ,

$$\sum_{j=0}^k \|\nabla^j u\|_{L^{\frac{2r}{k}}(\Omega)} \lesssim \|u\|_{L^\infty(\Omega)}^{1-\frac{k}{r}} \left( \sum_{i=0}^r \|\nabla^i u\|_{L^2(\Omega)} K^{r-i} \right)^{\frac{k}{r}}. \quad (\text{A.10})$$

### A.5 Interpolation on $\partial\Omega$

We need the following boundary interpolation inequalities to estimate the boundary part of our energy (1.16).

**Theorem A.6.** (Boundary interpolation) Let  $u$  be a  $(0, r)$  tensor, then

$$\|\bar{\nabla}^l u\|_{L^{\frac{2r}{k}}(\partial\Omega)} \lesssim \|u\|_{L^{\frac{2(r-l)}{k-l}}(\partial\Omega)}^{1-\frac{l}{r}} \|\bar{\nabla}^r u\|_{L^2(\partial\Omega)}^{\frac{l}{r}}. \quad (\text{A.11})$$

In particular, if  $k = l$ ,

$$\|\bar{\nabla}^k u\|_{L^{\frac{2r}{k}}(\partial\Omega)} \lesssim \|u\|_{L^\infty(\partial\Omega)}^{1-\frac{k}{r}} \|\bar{\nabla}^r u\|_{L^2(\partial\Omega)}^{\frac{k}{r}}. \quad (\text{A.12})$$

**Theorem A.7.** (Gagliardo-Nirenberg interpolation inequality) Let  $u$  be a  $(0, r)$  tensor, and suppose  $\partial\Omega \in \mathbb{R}^2$  and  $\frac{1}{l_0} \leq K$ , we have

$$\|u\|_{L^4(\partial\Omega)}^2 \lesssim_K \|u\|_{L^2(\partial\Omega)} \|u\|_{H^1(\partial\Omega)}, \quad (\text{A.13})$$

where the boundary Sobolev norm  $\|u\|_{H^1(\partial\Omega)}$  is defined via tangential derivative  $\bar{\nabla}$ .

*Proof.* It suffices for us to work in the local coordinate charts  $(U_i)_{i=1}^n$  of  $\partial\Omega$ . We introduce the corresponding partition of unity  $(\chi_i)_{i=1}^n$ , where each  $\chi_i$  is supported in  $U_i$  and vanishing on the boundary of  $U_i$ . Now by the result of Constantin and Seregin [2], we have

$$\|u_i\|_{L^4(U_i)}^2 \lesssim \|u_i\|_{L^2(U_i)} \|\bar{\nabla} u_i\|_{L^2(U_i)},$$

where  $u_i = \chi_i u$ . But since

$$\|\bar{\nabla} u_i\|_{L^2(U_i)} = \|\bar{\nabla}(\chi_i u)\|_{L^2(U_i)} \leq \|\bar{\nabla} \chi_i\|_{L^\infty} \|u\|_{L^2(U_i)} + \|\chi_i \bar{\nabla} u\|_{L^2(U_i)}, \quad (\text{A.14})$$

(A.13) follows by summing up (A.14) since  $\chi_i$  can be chosen so that

$$\sum_i |\nabla \chi_i| \leq C(K),$$

as long as  $\frac{1}{l_0} \leq K$  (see [1]). □

**Remark.** One can also prove a generalized (A.13) of the form

$$\|u\|_{L^p(\partial\Omega)}^2 \lesssim \|u\|_{L^{p/2}(\partial\Omega)} \|u\|_{H^1(\partial\Omega)}, \quad (\text{A.15})$$

for any  $p \geq 4$ .

Our next theorem shall be dealing with the projected derivatives acting on tensors. We first define that if  $\alpha$  is a  $(0, t)$  tensor, then the projected  $(0, r)$ ,  $r < t$  derivative  $\Pi^{r,0} \nabla^r \alpha$  has components

$$(\Pi \nabla^r)_{i_1, \dots, i_r} \alpha_{i_r+1, \dots, i_t} = \gamma_{i_1}^{j_1} \cdots \gamma_{i_r}^{j_r} \nabla_{j_1} \cdots \nabla_{j_r} \alpha_{i_r+1, \dots, i_t}.$$

The proof of the next theorem is rather involved, we refer [1] for the full proof.

**Theorem A.8.** (Tensor interpolation) Let  $\alpha$  be a  $(0, t)$  tensor, and let  $r' = r - 2$ , suppose that  $|\theta| + |\frac{1}{l_0}| \leq K$ , then for  $t + s < r$  we have

$$\begin{aligned} \|(\Pi^{s,0} \nabla^s) \alpha\|_{L^{\frac{2r'}{s}}(\partial\Omega)} &\lesssim_K \|\alpha\|_{L^\infty(\partial\Omega)}^{1-s/r'} \left( \|\nabla^{r'} \alpha\|_{L^2(\partial\Omega)} + (1 + \|\theta\|_{L^\infty(\partial\Omega)})^{r'} \right. \\ &\quad \cdot (\|\theta\|_{L^\infty(\partial\Omega)} + \|\bar{\nabla}^{r'} \theta\|_{L^2(\partial\Omega)}) \sum_{l=0}^{r'-1} \|\nabla^l \alpha\|_{L^2(\partial\Omega)} \Big) \\ &\quad + (1 + \|\theta\|_{L^\infty(\partial\Omega)})^s (\|\theta\|_{L^\infty(\partial\Omega)} + \|\bar{\nabla}^{r'} \theta\|_{L^2(\partial\Omega)})^{s/r'} \sum_{l=0}^{r'-1} \|\nabla^l \alpha\|_{L^2(\partial\Omega)}. \end{aligned} \quad (\text{A.16})$$

In particular,

$$\begin{aligned}
& \left\| (\Pi^{s,0} \nabla^s) \alpha \right\|_{L^2(\partial\Omega)} \left\| (\Pi^{r'-s,0} \nabla^{r'-s} \beta) \right\|_{L^2(\partial\Omega)} \lesssim_K \left( \|\alpha\|_{L^\infty(\partial\Omega)} + \sum_{l=0}^{r'-1} \|\nabla^l \alpha\|_{L^2(\partial\Omega)} \right) \|\nabla^{r'} \beta\|_{L^2(\partial\Omega)} \\
& + \left( \|\beta\|_{L^\infty(\partial\Omega)} + \sum_{l=0}^{r'-1} \|\nabla^l \beta\|_{L^2(\partial\Omega)} \right) \|\nabla^{r'} \alpha\|_{L^2(\partial\Omega)} + (1 + \|\theta\|_{L^\infty(\partial\Omega)})^{r'} (\|\theta\|_{L^\infty(\partial\Omega)} + \|\bar{\nabla}^{r'} \theta\|_{L^2(\partial\Omega)}) \\
& + \left( \|\alpha\|_{L^\infty(\partial\Omega)} + \sum_{l=0}^{r'-1} \|\nabla^l \alpha\|_{L^2(\partial\Omega)} \right) (\|\beta\|_{L^\infty(\partial\Omega)} + \sum_{l=0}^{r'-1} \|\nabla^l \beta\|_{L^2(\partial\Omega)}). \quad (\text{A.17})
\end{aligned}$$

*Proof.* See [1], section 4. □

## A.6 The trace theorem

**Theorem A.9.** (The trace theorem) Let  $\alpha$  be a  $(0, r)$  tensor, and assume that  $\|\theta\|_{L^\infty(\partial\Omega)} + \frac{1}{l_0} \leq K$ , then

$$\|\alpha\|_{L^2(\partial\Omega)} \lesssim_{K,r,n} \sum_{j \leq 1} \|\nabla^j \alpha\|_{L^2(\Omega)} \quad (\text{A.18})$$

*Proof.* Let  $\mathcal{N}$  be the extension of the normal to the interior, then the Green's identity yields

$$\int_{\partial\Omega} |\alpha|^2 d\mu_\gamma = \int_{\Omega} \nabla_k (\mathcal{N}^k |\alpha|^2) d\mu.$$

Hence, by Lemma A.1 and A.2, (A.18) follows. □

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